MATH 1370 - Assignment 2

1. **Population Variability**

   Consider a population of N neurons which on any given trial in a experiment produce the spike count vector $[y_1, \dots y_N]$. Let $\langle y_i \rangle = \mu$, $\text{Var}(y_i) = v$, and $\text{Cov}(y_i, y_j) = cv$ for all i and j (homogeneous statistics). Here $\langle \cdot \rangle$ denotes expectations across trials of the experiment. The parameter $0 \leq c \leq 1$ controls the amount of trial-to-trial co-variability across the population. Let the population activity drive a downstream target who sums over the population $Y = \frac{1}{N}\sum_{i=1}^{N} y_i$.

   (a) Show that the statistics of Y obey:

   $$\begin{aligned}\langle Y \rangle &= \mu, \\ \text{Var}(Y) &= v\left(\frac{1}{N} + c\frac{N-1}{N}\right).\end{aligned}$$

   (b) Consider the spike counts from a population of $N+1$ independent Poisson random variables, labeled $x_1, \dots, x_N, x_c$. Let the x_i's have identical statistics with intensity $\lambda = 100$ Hz and let x_c have intensity 5 Hz. Let each trial last for $T = 1$s. Construct a population of N correlated Poisson random variables with the mapping $y_i = x_i + x_c$. Determine μ, v, and c for the y_i's (you may use the results proved or stated in class about Poisson processes).

   (c) Simulate the above process for 500 trials and compute $\text{Var}(Y)$ [1]. Plot your numerical estimate of $\text{Var}(Y)$ against the theoretical prediction for $N = 1 \dots 200$.

2. **Synaptic balance and fluctuation driven activity**

   In this question you will explore how the variability of mixed excitatory and inhibitory synaptic inputs translates to the variability of the neural response.

   (a) Consider a mixed excitatory-inhibitory synaptic drive where the excitatory and inhibitory synaptic current is given by $I_E(t) = \bar{a}_E x_E(t)$ and $I_I(t) = \bar{a}_I x_I(t)$. A

[1] You may use the `MATLAB` command `poissrnd` to generate the random variables.

reasonable model for the time course of a single synaptic input is a difference of exponential terms:

$$x(t) = \left[e^{-\frac{t-\hat{t}}{\tau_1}} - e^{-\frac{t-\hat{t}}{\tau_2}}\right]\theta(t-\hat{t}),$$

where $\tau_1 > \tau_2$, $\theta(x)$ is a Heaviside function, and $\hat{t}$ is the time of a presynaptic spike. Show that a convenient differential equation treatment for this two-timescale synaptic current is given by:

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= -\frac{x}{\tau_1\tau_2} - \left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)y + \left(\frac{1}{\tau_2}-\frac{1}{\tau_1}\right)\sum_i \delta(t-t_i).\end{aligned}$$

and initial conditions $x(0) = y(0) = 0$. Here t_i is the time of the i^{th} presynaptic spike.

Let the sequences $\{t_{iE}\}_i$ and $\{t_{iI}\}_i$ be Poisson point process with distinct firing rates λ_E and λ_I. Find the mathematical expression relating $a_E > 0$, $a_I < 0$, $\lambda_E > 0$, and $\lambda_I > 0$ such that the mean[2] total current $\langle I(t)\rangle_t = \langle I_E(t)+I_I(t)\rangle_t = 0$, i.e is 'balanced'[3]. Use the `MATLAB` file `EIF_Stochastic_Synapse.m` to simulate $I(t)$ with $a_E = 1\mu\text{A/cm}^2$, $\lambda_E = \lambda_I = 500$ Hz, and a_I such that I is balanced. Let $\tau_{1E} = 4$ms, $\tau_{2E} = 0.4$ ms, $\tau_{1I} = 6$ ms, and $\tau_{2I} = 1.75$ ms. Use your simulation to verify that $\langle I\rangle_t \approx 0$.

(b) Consider the exponential integrate and fire neuron model[4] with current based synapses:

$$\begin{aligned}C\frac{dV}{dt} &= -g_L(V-V_L) + \phi(V) + I_E(t) + I_I(t),\\ I_E(t) &= a_E x_E(t),\\ I_I(t) &= a_I x_I(t),\end{aligned}$$

where $\phi(V) = g_L\Delta\exp\left(\frac{V-V_T}{\Delta}\right)$. Supplement the model dynamics with the spike-reset rule $V(t) = V_{peak}$ implies $V(t^+) = V_{reset}$. Let $C = 1\mu\text{F/cm}^2$, $g_L =$

[2] For a stochastic process we denote the mean as $\langle x(t)\rangle_t = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)dt$

[3] Hint: For a Poisson process we have $\langle y(t)\rangle_t = \lambda$. For a filtered Poisson process, say through a synapse, $\langle x(t)\rangle_t = \lambda\int_0^\infty x_1(t)dt$ where $x_1(t)$ is the synaptic response to a single input at time 0 (this is called Campbell's theorem).

[4] see Fourcaud-Trocmé N, Hansel D, van Vreeswijk C, Brunel N. How spike generation mechanisms determine the neuronal response to fluctuating inputs. J Neurosci. 2003, 23: 11628-40.

0.1mS/cm^2, $V_L = -65$ mV, $V_T = -60$ mV, $V_{reset} = -70$ mV, $V_{peak} = -45$ mV and $\Delta = 2$ mV. Let $x_E(t)$ and $x_I(t)$ be synaptically filtered inputs as in part (b) with $a_E = 0.2\mu\text{A/cm}^2$, $\lambda_E = 500$ Hz, and set inhibition to zero. Use the `MATLAB` file `EIF_Stochastic_Synapse.m` to simulate the above system and plot a 1s realization of $V(t)$ and $I(t)$. Run 1000 1s realizations and report the Fano factor $FF = \frac{Var}{M}$, where $M = \frac{1}{1000}\sum_i n_i$ and $Var = \frac{1}{1000}\sum_i (n_i - M)^2$, with n_i being the spike count in realization i.

(c) The firing rate for the inhibition free post-synaptic cell in c) should be about 13 Hz. Set $\lambda_E = \lambda_I = 500$ Hz and use the balance relation calculated in b) to determine a_I. This gives an effective line in (a_E, a_I) space. Explore this line to determine the point where the output firing rate from the postsynaptic cell is approximately 13 Hz. For these values of a_E and a_I compare a 1s realization of $V(t)$ and $I(t)$ to that in c). Compute the Fano factor and compare it to that in c).

3. **Renewal process**

(a) Consider a renewal process with hazard function $H(t) = \lambda\Theta(t - \tau_{\text{abs}})$. Here $\tau_{\text{abs}} >$ is the absolute refractory period and $\Theta(x)$ is the Heaviside function. Calculate:

i. the inter-spike interval probability density function $\rho(\tau)$.

ii. Show that the interspike interval coefficient of variance is

$$\text{CV} = \frac{\sqrt{\langle \tau^2 \rangle - \langle \tau \rangle^2}}{\langle \tau \rangle} = \frac{1}{1 + \lambda\tau_{\text{abs}}}.$$

(b) Write a MATLAB code to simulate the renewal process with Hazard function $H(t) = \lambda\left(1 - e^{-(t-\tau_{\text{abs}})/\tau_{\text{rel}}}\right)\Theta(t - \tau_{\text{abs}})$. Here $\tau_{\text{rel}} > 0$ is a relative refractory period. Let $\lambda = 30$Hz.

i. Numerically estimate $\rho(\tau)$ for $(\tau_{\text{abs}}, \tau_{\text{rel}}) =$ (3 ms,1 ms) and (3 ms, 10ms).

ii. Numerically estimate the CV as a function of τ_{rel} over (0,20ms) with $\tau_{\text{abs}} =$ 3ms.

iii. Numerically estimate the CV as a function of τ_{abs} over (0,20ms) with $\tau_{\text{rel}} =$ 3ms.