**MATH 1370 - Assignment 1**

1. **Quadratic Integrate and Fire Neuron**

   Consider the quadratic integrate-and-fire neuron model[1] (QIF) whose passive dynamics obey
   $$\tau \frac{dV}{dt} = (V - V_1)(V - V_2) + RI,$$
   where $\tau$ is the membrane time constant, $R$ is the input resistance, $I$ is an applied current, and $V_1, V_2$ are constants.

   (a) Consider the change of variables $\bar{V} = V - q$, $\bar{t} = pt$, and $\bar{I} = aI + b$ such that the passive equation becomes
   $$\frac{d\bar{V}}{d\bar{t}} = \bar{V}^2 + \bar{I}.$$
   Calculate $q$, $p$, $a$, and $b$ in terms of $\tau$, $R$, $I$, $V_1$, and $V_2$.

   (b) For $\bar{I} < 0$ determine the equilibrium points of $\bar{V}$ and evaluate their stability.

   (c) Show that for initial condition $V(0) = 0$ and $\bar{I} > 0$ we have the solution
   $$\bar{V}(\bar{t}) = \sqrt{\bar{I}} \tan\left(\sqrt{\bar{I}}\bar{t}\right).$$
   Conclude that $\bar{V} \to \infty$ in finite time.

   (d) For $\bar{I} > 0$ let $\bar{V} \to \infty$ represent a spike event, after which $\bar{V} \to -\infty$ is the reset. Let $T$ denote the time it takes for $\bar{V}$ to go from $-\infty$ to $\infty$ (i.e from reset to spike). Show that the firing rate is given by
   $$f = \frac{1}{T} = \frac{\sqrt{\bar{I}}}{\pi}, \qquad (\bar{I} \geq 0).$$

2. **Synaptic integration and coincidence detection**

   Consider a leaky integrate-and-fire neuron driven by a synaptic train. The membrane dynamics of the neuron obeys:

[1]Latham, Richmond, Nelson, Nirenburg. Intrinsic dynamics in neuronal networks. I. Theory. *J. Neurophysiology*, 83:808-827, 2000

$$C_m \frac{dV}{dt} + g_L(V - V_L) + g_{\text{syn}} y(t)(V - V_{\text{syn}}),$$

supplemented with the spike-reset rule $V(t) = V_T \implies V(t_+) = V_R$. Take the synaptic gating variable to be:

$$y(t) = \sum_{i=1}^{M} \left[ \exp\left( \frac{-(t - s_i)}{\tau_1} \right) - \exp\left( \frac{-(t - s_i)}{\tau_2} \right) \right] \theta(t - s_i).$$

The set $\{s_i\}$ gives times for synaptic discharge. The MATLAB code **LIFsynapse.m** simulates the above system for $M = 1$ and $s_1 = 10$. Standard parameter values are given in the code.

Modify **LIFsynapse.m** so that $M = 2$ with $s_1 = 10$ and $s_2 = 10 + \Delta$. For a fixed $g_{\text{syn}}$ define $\Delta_{\max}$ as the maximal value of $\Delta$ such that a spike response occurs[2]. Numerically range $g_{\text{syn}}$ over $(0.05, 0.095)$ and plot $\Delta_{\max}$.

3. **Planar reduction of Hodgkin-Huxley model**.

   The MATLAB code **HH.m** simulates the full Hodgkin-Huxley system for $\{V, m, h, n\}$. Consider the reduction outlined in class of the Hodgkin-Huxley model to a planar dynamical system $\{V, n\}$. Specifically, apply the approximations $m(t) \to m_\infty(V(t))$ and $h \to 0.8 - n$ (recall that $m_\infty(V) = \frac{\alpha_m(V)}{\alpha_m(V) + \beta_m(V)}$).

   (a) Modify **HH.m** to simulate the full Hodgkin-Huxley model and the reduced planar model. Compare simulations of the planar model to the full model for $I = 6$ and $t_{\max} = 10$ ms (let $V(0) = -70$ mV).

   (b) Plot the phase trajectory $(V(t), n(t))$ for the planar model for $I = 6$ and $I = 7$ (let $t_{\max} = 200$ ms). Discuss the differences in the two trajectories.

   (c) Determine the minimal current $I$ required to drive repetitive spiking for both the full Hodgkin-Huxley model and the reduced planar system. Discuss why these currents may not be equal to one another.

[2] $\Delta_{\max}$ does not exist for all $g_{sun}$ values