This is a long HW problem that involves a combination of simulations and also of analysis. We will look a model of all-to-all coupled quadratic integrate-and-fire neurons:

\[ V'_j = V^2_j + \mu + \Delta L_j + ks, \quad j = 1, \ldots, N \]

with the rule that when \( V_j = +\infty \), it is reset to \(-\infty\). We call the time at which \( V_j \) hits \(+\infty\), \( t_j \). The random numbers, \( L_j \) are taken from the Lorenz distribution (which we generate as)

\[ L_j = \tan(\pi (R_j - 1/2)) \]

where \( R_j \) are uniform random numbers on \((0, 1)\). The synaptic coupling obeys the ODE:

\[ s' = -s/\tau + (1/N) \sum_{j=1}^{N} \delta(t - t_j)/\tau. \]

Note that the sum in this equation is the average firing rate of the network. So, the first thing I want you to do is simulate the equations for \( N = 400 \) and “infinity” (the reset value) set to, say, 100. Set \( \tau = 0.5 \), \( \mu = 5 \), \( \Delta = 0.25 \). Use Euler to integrate with \( dt=0.005 \) and run for 40000 iterations (200 time units) with a variety of values of \( g \), say \( k = -3, -2, -1, 0, 1, 2, 3 \) and plot \( s \) vs time; maybe zoom in to see if there is a rhythm at all. For \( k \) in the right range you should see some regular oscillations. I will post some XPP code if you don’t want to try this using MatLab, Python, or something else (like Julia).

So now we need to analyze this. Proceeding as we did in class with the Kuramoto, we will let \( I_j = \mu + \Delta L_j \) and \( N \to \infty \). We let \( f(V, I, t) \) be the density of \( V, I \) at any point \( t \) and find that

\[ \partial_t f + \partial_V [(V^2 + I + ks)f] = 0. \] (1)

The firing rate is the flux at \( V = +\infty \), that is

\[ r(I, t) = \lim_{V \to \infty} (V^2 + I + ks)f(V, I, t) \]

Let \( g(I) \) be the density for the applied currents. Then the average firing rate, \( r_a(t) \) is

\[ r_a(t) = \int_{-\infty}^{\infty} g(I)r(I, t) \, dI. \]

We also see that by this definition that \( s \) satisfies

\[ \tau s' = -s + r_a(t) \]

which now closes the system.

Now it is time for an ansatz. We suppose that

\[ f(V, I, t) = \frac{1}{\pi} \frac{\beta(I, t)}{(V - \alpha(V, t)) + \beta^2(V, t)}. \]
Using this \textit{ansatz} derive equations for $\alpha, \beta$. Plug the ansatz into Eq (1) and you will get something that looks like:

$$\frac{(AV^2 + BV + C)}{M(V)} = 0$$

This must be true for all $V$, so $A, B, C$ must vanish. $M$ is complicated, but we don’t care since it is in the denominator and get rid of it. Solve $A$ for $\beta_t$ and you will get

$$\beta_t = 2\beta\alpha$$

and plug this into, say, $C$ and you will find

$$\alpha_t = I + ks + \alpha^2 - \beta^2.$$  

Finally, plug these into $B$ to make sure it vanishes! (It will). Now we are cooking. Let’s do one more thing. Let $w = \alpha + i\beta$. Show that

$$\frac{\partial w}{\partial t} = w^2 + I + ks$$

Now, lets close this sucker up. We note that $r(I, t) = \beta/\pi$ so that

$$\pi r_a(t) = \int_{-\infty}^{\infty} g(I)\beta(I, t) \, dI$$

so that we can now get an ODE for $s$. Also note that the average voltage is

$$V_a(t) = \int_{-\infty}^{\infty} g(I)\alpha(I, t) \, dI.$$  

As with the Kuramoto done in class, we still have an infinite dimensional system. I will write it as

$$\frac{\partial w}{\partial t} = w^2 + \mu + \Delta L + ks$$

and parametrize it by $L$ instead which is just taken from the Lorenzian in our simulations. That is, instead of $w(I, t)$, I have $w(L, t)$. We now see that

$$V_a(t) + i\pi r_a(t) = \int_{-\infty}^{\infty} g(L)w(L, t).$$

So, let’s suppose that $g(L) = 1/(\pi[1 + L^2])$ and as in class use the residue theorem to evaluate the integral. We write

$$g(L) = \frac{1}{2\pi i} \left( \frac{1}{L - i} - \frac{1}{L + i} \right),$$

so that

$$V_a(t) + i\pi r_a(t) = w(\pm i, t)$$
depending on which of the two possible contours you take. (Note the $2\pi i$ from the residue theorem cancels with the $2\pi i$ in the denominator.) This yields:

$$\frac{dw(\pm i, t)}{dt} = w(\pm i, t)^2 + \mu + ks \pm i\Delta.$$ 

Justify why you must take the $+i$ root, recalling that $\beta$ must be positive. This leads to the following ODEs that constitute the system to analyze:

$$a' = \mu + ks + a^2 - b^2$$

$$b' = 2ab + \Delta$$

$$\tau s' = -s + b/\pi.$$ 

Here, $a = \alpha(+i, t), b = \beta(+i, t)$ as required. Notice that $a$ is the average potential and $b/\pi$ is the firing rate. Do the following:

1. Show that if $b(0) > 0$, then $b(t) > 0$ for all time as long as $\Delta > 0$.
2. Show that equilibria require that $a < 0$ since $b > 0$. Conclude that the average potential must be negative.
3. The uncoupled system has $k = 0$. Find the firing rate $F(\mu)b/\pi$ as a function of $\mu$ and sketch it for $\Delta = 0.1$ and $\mu \in (-2, 2)$. Show that $F(\mu)$ approaches $\sqrt{\mu}/\pi$ which is the single cell QIF firing rate curve.
4. For $k \neq 0$ explore the equilibria by setting $s = b/\pi$ and then exploring the $(a, b)$ phase plane. Try to prove that there are at most 3 equilibria with $b > 0$ and always at least 1. Hint: Note the $b$ nullcline asymptotes at the axes and that the $a$ nullcline can be written as a hyperbola:

$$(b - K/2)^2 - a^2 = \mu + K^2/4$$

where $K = k/\pi$. Recall from Calc I how to sketch hyperbolas!
5. Now, back to the full model (that is including $s$) This might be hard. Can you show that there are no Hopf bifurcations if $k > 0$? Show that there is only one $b > 0$ fixed point when $k < 0$
6. Sketch the bifurcation diagram for $k = 4, \tau = 0.5, \Delta = 0.1$ as the drive, $\mu$ to the network increases between $-0.5$ to 0.5.
7. Set $k = -3, \tau = 0.5, \Delta = 0.1$ and compute the bifurcation diagram as $\mu$ increases from -1 to 5. If there is a Hopf bifurcation, follow the periodic orbit. For $\mu = 5, k = -3$ does the behavior of the simple system appear to agree with the full blown coupled QIF you solved above?