Waves, waves, waves

We are interested in finding traveling waves to the following equation:

$$\frac{\partial V}{\partial t} = f(V,n) + \frac{\partial^V}{\partial x^2}, \quad \frac{\partial n}{\partial t} = \epsilon g(V,n)$$

when $\epsilon$ is small and positive. Propagating pulses are solutions, $(V(x,t), n(x,t)) = (u(\xi), w(\xi))$ to the following differential equation:

$$-\theta u' = f(u,w) + u'', \quad -\theta w' = \epsilon g(u,w) \tag{1}$$

where the primes mean differentiation with respect to $\xi = x - \theta t$, the moving coordinate. We want solutions which start at rest at $+\infty$, spike, and return to rest at $-\infty$. In general, this is a very difficult problem to solve. Suppose that $\epsilon$ is small. Setting $\epsilon = 0$, means that $w$ is essentially constant, so this reduces to a second order equation which we analyze below. I introduce a scaled coordinate, $\eta = \epsilon\xi$ so that we get

$$-\epsilon\theta U_\eta = f(U,W) + \epsilon^2 U_{\eta\eta}, \quad -\theta W_\eta = g(U,W). \tag{2}$$

Setting $\epsilon = 0$, leads to $f(U,W) = 0$. $f$ is a cubic like function, so that for any given $W$, there can be up to three roots.

Basically, the strategy is to construct the wave in four parts: (I) resting to the excited state holding $w$ constant and using equation (1); (II) ride along the "up" state using equation (2), with $U = U^+(W)$ where $U^+$ is the maximal (in $U$) root of $f(U,W) = 0$; (III) jump back to the low $U$ state at a sepcified value of $w$ using equation (1) again; (IV) follow equation (2) using $U = U^-(W)$ (the lowest root of $f(U,W) = 0$ back to rest.

To do this, we first explore wavefronts in the nonlinear PDE:

$$V_t = f(V) + V_{xx}$$

obtained by holding $n$ constant. We look for solutions to this of the form $V(x,t) = u(x - \theta t)$ where $\theta$ is the velocity of the waves. We suppose that $f(u)$ has three roots, $u_1 < u_2 < u_3$ and that $u_1, u_3$ are stable fixed points to $u_t = f(u)$. Let $\xi = x - \theta t$ so that $u(\xi)$ satisfies:

$$u'' + f(u) + \theta u' = 0. \tag{3}$$

We seek solutions satisfying $u(-\infty) = u_3$ and $u(+\infty) = u_1$. That is, a front joining the two stable fixed points. We can write this as a system

$$u' = z, \qquad z' = -f(u) - \theta z$$

and you can verify that $(u_1, 0), (u_3, 0)$ are saddle-points and $(u_2, 0)$ is a node (or center if $\theta = 0$.). Multiplying equation (3) by $u'$ and integrating over $\xi$ from $-\infty$ to $\infty$, we find that

$$\theta = K \int_{u_1}^{u_3} f(u)du.$$

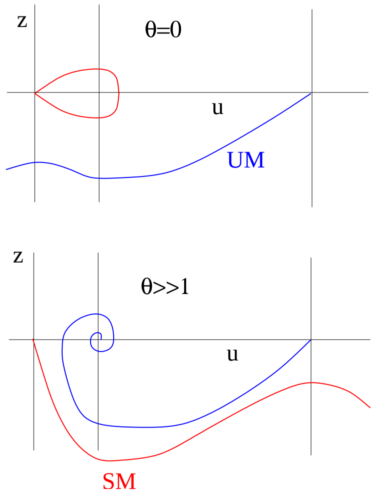


Figure 1: Diagram of the behavior of the stable manifold and instable manifold of the two fixed points as $\theta$ varies.

where $K$ is a positive constant, so that the sign of the wave velocity is the same as the sign of the area of $f(u)$ between $u_1$ and $u_3$. Henceforth, we assume this area is positive so that the $u_3$ state takes over the medium; the velocity is positive. Since $u_3$ is a saddle-point, it has an unstable manifold (UM). We want to follow the lower left branch of the manifold from $u_3$ to the root $u_1$. For $\theta = 0$, we have

$$u'' = -f(u)$$

which implies that

$$E \equiv z^2/2 + F(u)$$

where $F = \int f(u)du$. Verify for yourselves that $dE/d\xi = 0$ on solutions. You should also show that

$$\frac{dE}{d\xi} = -\theta \left(\frac{du}{d\xi}\right)^2.$$

Think of $E$ as the energy. For $\theta = 0$, $E$ is constant along trajectories, so that it is easy to show that UM hits the line $u = u_1$ for $z < 0$. Consider the stable manifold (SM) coming out of $u_1$ in the negative $z$ quadrant. For $\theta = 0$, it is homoclinic, looping around $(u_2, 0)$. For $\theta$ small enough, it crosses the $z$-axis and winds past $(u_2, 0)$ before heading off to infinity in the upper left direction. For $\theta$ larger SM goes deeper into the $z < 0$ quadrant. Recall, that we track SM by going backwards in $\xi$. Thus, if we push SM far enough out (that is make $\theta$ big), the function $E$ grows since $dE/d\xi > 0$ (in the negative $\xi$ direction), which drives SM past the line $u = u_3$ in the negative $z$ half-plane. Thus for small $\theta$, UM goes below SM and for large $\theta$, UM goes above SM. Thus, by continuity, there is a value of $\theta$ such that they coincide and this is the solution we desire. I have illustrated this for the sodium model. Only the UM is shown. Here is an

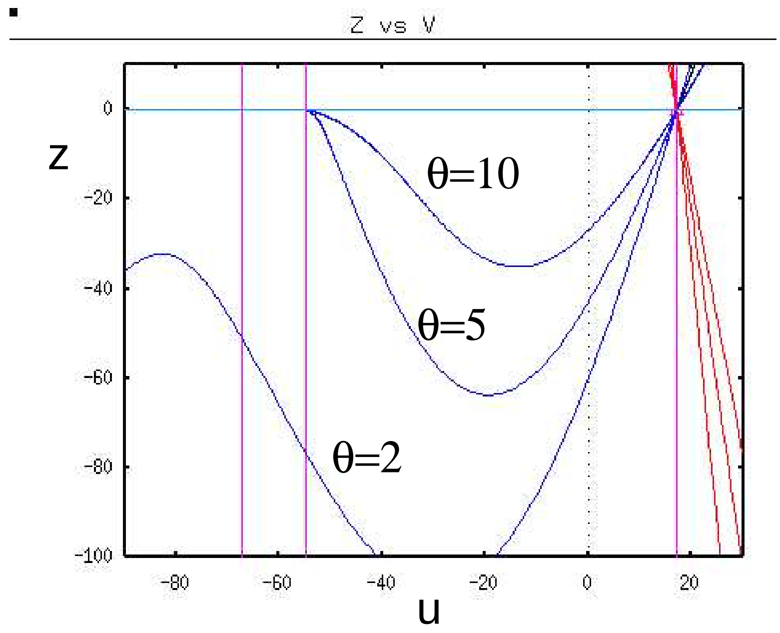


Figure 2: Several examples of the UM from $(u_3, 0)$ for different values of $\theta$. We want to hit $(u_1, 0)$, so we should pick $\theta$ somewhere between 2 and 5.

XPP file for the sodium model with potassium held at rest:

```
ion(v,n)=(I-gl*(V-el)-gk*n*(v-ek)-gna*minf(v)*(v-ena))/C
minf(v)=1/(1+exp(-(v-va)/vb))
tau(v)=tau0
v'=z
z'=-b*z-ion(v,n)
par b=0
par I=0,n=.0115,vc=-45,tau0=5,gl=8,gk=10,gna=20,el=-80,ek=-90,ena=60
par va=-20,vb=15,vd=5
par c=1
init v=17.2,z=0
@ meth=rk4,dt=.01,total=15,xp=v,yp=z,xlo=-90,xhi=30,ylo=-100,yhi=10
@ bound=1000
done
```

Here is how to shoot!

1. Set the initial data close to $(u_3, 0)$. This is done for you already.
2. Set $\theta$ to a guessed value. (Here it is inexplicably called `b`.)
3. Click on SingPts Go. Answer `N` for `print eigenvalues` and `Yes` to `Draw Invariant sets.` Hit the `Enter` key when the `Out of Bounds` happens

- this is the upper branch of UM. The lower branch (yellow) will either go out of bounds or get sucked into $(u_2, 0)$. In the latter case, hit `ESC` to stop the integration. For velocities that are too low, you head to infinity and for too high, you get sucked into $(u_2, 0)$. So by using bisection, you should be able to converge to a decimal place or so of the correct velocity.

Let me try to describe the singular construction of the traveling wave. This is illustrated in the figure below. We start at $(u_{rest}, w_{rest})$ which sets the level of the recovery variable. The function $f$ has three roots, one of which is $u_{rest}$ and the other is $U^+(w_{rest})$. We have just found there is a wave joining these two fixed points which uniquely determines the velocity. The wave velocity is parametrized by the constant level of $w$ the recovery variable. In other words for each $w$ is some interval, there is a traveling front to

$$-\theta(n)u' = f(u, w) + u''.$$

As $w$ increases the wave slows down and for large enough $w$, reverses course. We choose $w_{jump}$ so that

$$\theta(w_{rest}) = -\theta(w_{jump}).$$

This uniquely determines $w_{jump}$. As before $f(u, w_{jump})$ has three roots, and we want a jump between the right-most one, $U^+(w_{jump})$ and the left-most one, $U^-(w_{jump})$. This construction determines the two fast regions (which are horizontal in the upper diagram) I,III. How do we get from $U^+(w_{rest})$ to $U^+(w_{jump})$? For this, we go to equation (2) when $\epsilon = 0$. Since we are on the right branch and nned to solve $f(U, W) = 0$, we take $U = U^+(W)$:

$$-\theta W_\eta = g(U^+(W), W) \tag{4}$$

We solve (4) with initial data $W(0) = w_{rest}$ and integrate backwards in time $\eta$ decreasing, to reach $W = w_{jump}$ which has already been determined. Note that $g(U^+(W), W) > 0$ so that this can be done. Once $W$ reaches $w_{jump}$, we make the jump to the left branch along solution III. Now we recover with:

$$-\theta W_\eta = g(U^-(W), W) \tag{5}$$

with $W(0) = w_{jump}$. Note that $g(U^-(w_{rest}), w_{rest}) = 0$ so that $w_{rest}$ is a fixed point of (5). Thus, $W(\eta)$ will decay to the rest state dragging $U$ with it through the relationship, $U = U^-(W)$. Thats it! There are really only two unknowns - the jump-up velocity, found by shooting and the value of the recover at which we jump back, found by matching the jump-up velocity, but with opposite sign.

I include here for your amusement, an XPP file for the full discretized model:

```
#
ion(v,n)=(I-gl*(V-el)-gk*n*(v-ek)-gna*minf(v)*(v-ena))/C
minf(v)=1/(1+exp(-(v-va)/vb))
ninf(v)=1/(1+exp(-(v-vc)/vd))
```

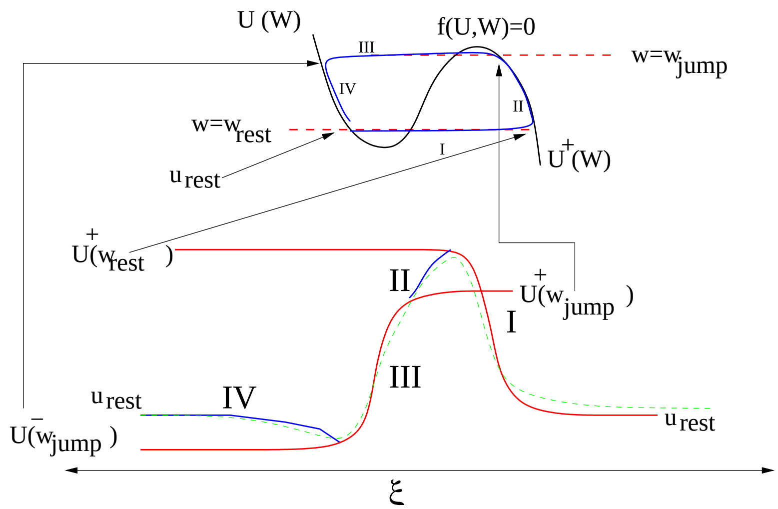


Figure 3: Singular construction of the wave.

```
tau(v)=tau0
par d=1
par I=0,vc=-45,tau0=5,gl=8,gk=10,gna=20,el=-80,ek=-90,ena=60
par va=-20,vb=15,vd=5
par c=1
v0=v1
v51=v50
v[1..50]'=ion(v[j],n[j])+d*(v[j-1]-2*v[j]+v[j+1])
n[1..50]'=(ninf(v[j])-n[j])/tau(v[j])
init v[1..3]=-50,n[j]=.0115
init v[4..50]=-67.267,n[j]=.0115
@ total=30,xhi=30,ylo=-80,yhi=20,yp=v25,nplot=2,yp2=v35
@ meth=rk4,dt=.02
done
```

It is set up to show the voltage at 25 and 35. The time difference between these should give you an approximation of the velocity. Compare it to the singular velocity.

**HOMEWORK.**

1. Compute $\theta$ (`b`) to one decimal place for the sodium model.
2. Compare it to the velocity for the full discrete model. Note this is a crude discretization, so that the best you can hope for is to be within an order of magnitude.
3. Find $\theta$ exactly for the following model:

$$u_t = f(u) + u_{xx}$$

where $f(u) = -u + H(u - a)$ where $H$ is the unit step-function and $a \in (0, 1)$. Note that 0 and 1 are the stable fixed points of $f$. You will want to solve this for $u < a$ and for $u > a$ as two pieces and join them together continuously and differentiably.

4. Here is a cool trick. Consider

$$f(u) = Au(u - a)(1 - u)$$

where $a \in (0, 1)$. This has three fixed points, with 0 and 1 stable. We want to solve

$$-\theta u' = f(u) + u''$$

with $u(-\infty) = 1$ and $u(+\infty) = 0$. Consider the equation

$$u' = -bu(1 - u)$$

Find a value of $b$ and $\theta$ so that the solution to the second equation is a solution to the first. Thus, find an exact expression for the velocity. Solve the second equation by quadrature to get a closed form expression for ther wave front!

5. **Difficult!** In the above exercises, we found a front joining two fixed points. Suppose, that pour phase-space is a ring instead of the line, say, for simplicity that $f(u) = I - \cos u$ where $0 < I < 1$. Consider

$$u_t = f(u) + u_{xx}.$$

$f(u)$ has fixed points, $u_1, u_2$ and multiples of $2\pi$ added to these. In particular, if we look at the interval $[u_1, u_1 + 2\pi)$, then as each of these is a stable fixed point to $f(u)$, there is a front solution for the PDE system. But, since on the circle, $u_1 = u_1 + 2\pi$, the front is a pulse! This is the simplest model to generate a traveing pulse.

   (a) Numerically find the velocity of the pulse for $I = .9$.

   (b) Excitable systems have a one-parameter family of periodic solutions as well as the solitary pulse. These are solutions of the form $U(x-\theta t)$ with $U(\xi)$ periodic with period $P$ in $\xi$. Note that $\theta$ is different for each $P$ and converges to the traveling pulse equation as $P \to \infty$. Consider the following equation on a circle:

$$-\theta u' = f(u) + u''$$

   and show that there exists a one-parameter family of solutions, $U(\xi; \theta)$ such that $U(P; \theta) = U(0; \theta) - 2\pi$ and $z(0) = z(P)$. The figure illustrates the concept. The red line is the traveling wave. The black curve is the nullcline in $(u, u' = z)$ space. The blue line is an example of the kind of periodic solution that I am interested in. Note that $\theta$ will be different for each value of $P$. If you need some hints, stop by and I will give them to you. Note that the vector field is periodic in $u$.