Homework # 6

1. An activator-inhibitor system at a Hopf bifurcation has linearization:

$$M = \begin{pmatrix} a & -b \\ c & -a \end{pmatrix}$$

where a, b, c are positive parameters. Suppose that $bc > a^2$. Compute the eigenvector, Φ corresponding to the imaginary eigenvalue, $i\omega = i\sqrt{bc - a^2}$. Compute the adjoint eigenvector Ψ where $M^T\Psi = -i\omega\Psi$ and $\bar{\Psi}^T\Phi = 1$. The coupling matrix is:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Write down the quantity, $\beta = \bar{\Psi}^T C\Phi$ and show that β has a positive real part.

2. Recall that a pair of coupled Hopf bifurcations satisfies:

$$\begin{aligned} z_1' &= z_1(1 + (-1 + iq)|z_1|^2) + (\mu + i\gamma)(z_2 - z_1) \\ z_2' &= z_2(1 + (-1 + iq)|z_2|^2) + (\mu + i\gamma)(z_1 - z_2) \end{aligned}$$

Let $z_j = r_j \exp(i\theta_j)$ and let $\phi = \theta_2 - \theta_1$. Show that

$$\begin{aligned} r_1' &= r_1 - r_1^3 + \mu(r_2 \cos\phi - r_1) - \gamma r_2 \sin\phi \\ r_2' &= r_2 - r_2^3 + \mu(r_1 \cos\phi - r_2) + \gamma r_1 \sin\phi \\ \phi' &= q(r_2^2 - r_1^2) - \mu\left(\frac{r_2}{r_1} + \frac{r_1}{r_2}\right)\sin\phi + \gamma\left(\frac{r_1}{r_2} - \frac{r_2}{r_1}\right)\cos\phi \end{aligned}$$

Find conditions for which there is a solution of the form, $r_1 = r_2 = R$ and $\phi = \pi$. Suppose that $q = 2$ and $\gamma = 1$. Use the computer to explore the behavior as a function of μ starting with μ near zero and going up to 1.

3. Consider a ring of coupled Hopf bifurcations:

$$z_j' = z_j[1 + (-1 + iq)|z_j|^2] + (\mu + i\gamma)(z_{j+1} - 2z_j + z_{j-1}), \quad j = 1, \ldots, N$$

with $z_0 \equiv z_N$ and $z_{N+1} \equiv z_1$. Find conditions for which there exists a traveling wave, $z_j = R_N \exp(i\Omega_N + 2\pi j/N)$ and find expressions for R_N, Ω_N.

4. Consider the chain:

$$\begin{aligned} \theta_1' &= \omega + \sin(\theta_2 - \theta_1 - \alpha) \\ \theta_j' &= \omega + \sin(\theta_{j+1} - \theta_j - \alpha) + \sin(\theta_{j-1} - \theta_j + \alpha) \\ \theta_N' &= \omega + \sin(\theta_{N-1} - \theta_N + \alpha) \end{aligned}$$

Find a traveling wave for this, $\theta_j = \Omega t + cj$ and determine it's stability.

5. Consider the weakly coupled pair of oscillators:

$$\begin{aligned} \theta_1' &= \omega + \epsilon P(\theta_2)\Delta(\theta_1) \\ \theta_2' &= \omega + \epsilon P(\theta_1)\Delta(\theta_2) \end{aligned}$$

Let $\theta_j = \psi_j + \omega t$ and note that

$$\begin{aligned} \psi_1' &= \epsilon P(\omega t + \psi_2)\Delta(\omega t + \psi_1) \\ \psi_2' &= \epsilon P(\omega t + \psi_1)\Delta(\omega t + \psi_2) \end{aligned}$$

which we can average, yielding

$$\begin{aligned} \psi_1' &= \epsilon H(\psi_2 - \psi_1) \\ \psi_2' &= \epsilon H(\psi_2 - \psi_2) \\ H(\phi) &= \int_0^1 P(s+\phi)\Delta(s)ds \end{aligned}$$

Letting $\phi = \psi_2 - \psi_1$, we get

$$\phi' = H(-\phi) - H(\phi)$$

Suppose that $P(\theta) = 0$ for $|\theta - \theta_T| > \sigma$ and $P(\theta) = 1/(2\sigma)$ for $|\theta - \theta_T| < \sigma$ (of course modulo 1,so that if $\theta_T = 0$, then P vanishes in the interval $(\sigma, 1-\sigma)$). Assume that $\sigma < 1/2$. Determine all the phase-locked states (equilibria of the ϕ equation), when $\Delta(\theta) = -\sin 2\pi\theta$ and $\Delta(\theta) = 1 - \cos 2\pi\theta$. Note that these correspond respectively to the adjoints near a Hopf and a saddle-node.

6. Obtain the complete bifurcation diagram for the self-coupled excitatory neuron model without saturation:

$$\frac{ds}{dt} = \alpha_0\sqrt{g(s-s^*)} - s$$

where the term inside the square root is set to zero if $s < s^*$. Assume, s^*, α_0, g are all positive.