Homework #3

1. The quadratic integrate and fire model can have squarewave bursts in some parameter regimes. Consider:

$$V' = V^2 + a - W, \qquad W' = \epsilon W$$

where when $V = 10$, V is reset to 1 and W is incremented by d. For what range of (a, d) does fast-slow analysis predict bursting. Here is how to proceed. Treat W as a parameter and look at the V equation. Figure out where the "homoclinic" is and when there will be repetitive spiking. Figure out how big W has to get to stop the spiking. Then you can estimate how long it will take W to decay enough to start spiking again. Use this to get an estimate of a. Note that d will mostly control how many spikes per burst. If you set ϵ small enough, you can pretend that W does not decay at all between spikes and from this figure out how many spikes per burst as a function of a, d. You may want to simulate this to check your theory.

2. As I mentioned in class, you can create a 1-dimensional map for the burster.The map is piecewise defined and I have found it to be

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f1(x)=a1+b1*x
f2(x)=f1(x1)-b2*(x-x1)
f3(x)=f2(x2)
a1=.2,b1=.8,b2=15, x1=1.3,x2=1.36
f(x)=if(x<x1)then(f1(x))else(if(x<x2)then(f2(x))else(f3(x)))
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The map is defined as:

$$x_{n+1} = f(x_n) + i$$

where i is the parameter. A fixed point is defined as $x = f(x)$ and it will be stable if $|f'(x)| < 1$. A fixed point corresponds to a tonic spiking solution in the burster if it occurs on the first part of f. That is, it is not bursting. Find the maximum value of i so that there is a tonic spiking solution. Note that the slope of f is 0.8 on the first par, -15 on the second part, and 0 on the third part. You should be able to do this analytically since you just have to solve a linear equation. Through simulation, try to find some periodic solutions, that is, $x_{n+M} = x_n$ for $M > 1$. For example, if you pick $i = .15$, you should find a period 11 solution!. Try to get period 5,4,and 3. At what value of i does spiking cease? Hint: this coresponds to a fixed point on the flat part of f $(x > x_2)$.

3. For the elliptic burster, we consider the simple model. If we ignore the θ variable, then:

$$\begin{aligned} r' &= r(p + r - r^2) \\ p' &= \epsilon(r_0 - r) \end{aligned}$$

Simulate this simple model for $r_0 = 0.5$ and $\epsilon = .05$. You will see an oscillation in r if you start with initial conditions, $r(0) > 0$ and, say, $p = 0$. This corrsponds to bursting since the regions where r is near zero are silent and those where r is close to 1 are spiking. Regular spiking corresponds to an equilibrium where $r > 0$. Find this equilibrium and its stability. For what values of ϵ, r_0 will there be a Hopf bifurcation. Compute the period in the singular limit; it involves computing this integral:

$$f(r_0) = \int_0^{-1/4} dp/(r_0 - (1/2 + \sqrt{(}1/4 + p)))$$

(Maple make a mess of this, but it is really not so bad. REDUCE works much better!) If you want, try to compute the singular period in terms of r_0 given this integral. (The key here is showing that this integral actually shows up!).

4. Now onto parabolic bursting. Consider:

$$\begin{aligned} u' &= 1 - \cos(u) + (1 + \cos(u))(a + b * x - c * y) \\ x' &= \epsilon(-x + \delta(u - \pi)) \\ y' &= \epsilon(-y + \delta(u - \pi))/\tau \end{aligned}$$

where by $\delta(u - \pi)$ we mean that each time u crosses π, increment by 1. If you fix x, y, show that the frequency of u (That is, the period is the time it take u to go from $-\pi$ to π is $1/f$) is $f = \sqrt{(}\max(a + bx - cy, 0))$. The average of $\delta(u - \pi)$ is exactly f. Thus, the averaged (x, y) system is:

$$\begin{aligned} x' &= \epsilon(-x + f) \\ y' &= \epsilon(-y + f)/\tau \end{aligned}$$

Show via nullclines, simulation, etc, that there are some values of a, b, c, τ where the above system has a limit cycle. Note by rescaling time in the above, you can set $\epsilon = 1$.

5. Compute the velocity of the wavefront in the piecewise linear model:

$$V_t = f(V) + V_{xx}$$

where $f(V) = -V + H(V - \theta)$ and H is the step function. Proceed as follows. First the traveling system is

$$-cV' = f(V) + V''$$

Note that $f(0) = f(1) = 0$ when $0 < \theta < 1$. You want a solution that satisfies $V(-\infty) = 1$ and $V(\infty) = 0$. Note that the equation is always linear except at the jump. Finally, since the wave is translation invariant, you should choose coordinates so that $V(0) = \theta$. So, $V(\xi) > \theta$ for $\xi < 0$ and $V(\xi) < \theta$ for $\xi > 0$.

6. Using the results of the above exercise, compute the *singular* solution to the Rinzel pulse model:

$$\begin{aligned} -cV' &= f(V) - w + V'' \\ -cw' &= \epsilon[V - bw] \end{aligned}$$

Note that you should make sure b is chosen so that there is only the equilibrium $(0,0)$. Proceed as follows. When $\epsilon = 0$ start at rest, so $w = 0$ and use the previous exercise to jump up to the $V > \theta$ branch. Now rescale ξ and set $\epsilon = 0$ to get

$$\begin{aligned} 0 &= f(V) - w \\ -cw' &= V^+(w) - bw \end{aligned}$$

where you need to solve for $V^+(w)$. This is a linear equation in w starting with $w(0) = 0$ so you have to integrate it until $w = w_{jump}$ Figure out w_{jump} from the previous problem since it requires that the velocity be opposite the jump up velocity. Last but not least, compute the solution to the w' equation with V^+ replaced by V^-, the low V root of $f(V) - w = 0$. An interesting extension of this is to try to compute the singular periodic orbits that happen for a lower value of c than the singular homoclinic. Here is how to do this. Pick a small value of w, call it w_1 that is near zero. There will be 2 roots to $f(V) - w_1 = 0$, so choose $c(w_1)$ to create a wave that jumps from one to the other. Now on the right branch $V^+(w)$, solve the w system. Figure out the place to jump back (w_2) to the V^- side by matching the velocity to your jump up velocity. Then solve the w equation on V^- until w again equals w_1. The total transit time is the period, T. Both c and T are parametrized by w_1, so you should be able to compute the singular dispersion relation!

7. Problem 1,6,7 Chapter 6.