

## 1 Regular series

We will study the solutions to certain second order linear differential equations. We will start with equations that are analytic at  $t = 0$  and then turn to those that have singularities. Recall that a function is *analytic* at a point  $t = t_0$  if it has a convergent Taylor series at that point. We are interested first in equations of the form:

$$x'' + p(t)x' + q(t)x = 0 \quad (1)$$

where  $p(t), q(t)$  have Taylor series at  $t = 0$ . The basic idea is to assume that there is a solution of the form:

$$x(t) = a_0 + a_1t + a_2t^2 + \dots$$

and then try to prove that the series converges. In general, we will get an iterative series of equations for  $a_j$  in terms of already known coefficients. The first two coefficients will be determined by the initial conditions:

$$x(0) = a_0 \quad x'(0) = a_1.$$

Lets do a simple example that we already can solve:

$$x'' + x = 0$$

Substituting the series into the derivative:

$$x''(t) = 2a_2 + 6a_3t + 12a_4t^2 + \dots + j(j-1)t^{j-2}a_j +$$

and equating the coefficients we get

$$\begin{aligned} 0 &= a_0 + 2a_2 \\ 0 &= a_1 + 6a_3 \\ 0 &= a_2 + 12a_4 \\ 0 &= a_{2j-1} + (2j+1)(2j)a_{2j+1} \\ 0 &= a_{2j} + (2j+2)(2j+1)a_{2j+2} \end{aligned}$$

Thus we get a recursive relationship for the coefficients:

$$\begin{aligned} a_{2j+2} &= -\frac{1}{(2j+2)(2j+1)}a_{2j} \\ a_{2j+1} &= -\frac{1}{(2j+1)(2j)}a_{2j-1} \end{aligned}$$

From this it is easy to see that

$$a_{2j+1} = \frac{(-1)^j}{(2j+1)!}a_1$$

and

$$a_{2j} = \frac{(-1)^j}{(2j)!}a_0$$

The series breaks into an even part that depends only on  $a_0$  and an odd part that depends only on  $a_1$ . Writing the two series down, we immediately recognize them:

$$x_{\text{even}}(t) = a_0(1 - t^2/(2!) + t^4/(4!)) \dots = a_0 \cos t$$

and

$$x_{\text{odd}}(t) = a_1(t - t^3/(3!) + t^5/(5!) \dots) = a_1 \sin t.$$

Thus, we recover the already known solutions,  $\sin(t), \cos(t)$ .

Let's try an example that we don't already know:

$$x'' - tx = 0$$

Once again we expand in a series and obtain

$$\begin{aligned} 2a_2 &= 0 \\ 6a_3 - a_0 &= 0 \\ 12a_4 - a_1 &= 0 \\ 20a_5 - a_2 &= 0 \\ j(j-1)a_j - a_{j-3} &= 0 \end{aligned}$$

This implies that

$$a_j = \frac{1}{j(j-1)} a_{j-3}.$$

Note that  $a_2 = a_5 = a_8 \dots = 0$ ,  $a_3, a_6, \dots, a_{3j}$  depend on  $a_0$  and  $a_4, a_7, \dots, a_{3j+1}$  depend on  $a_1$ . Note that the series converges for all  $t$  since

$$\left| \frac{a_{j+3}t^{j+3}}{a_j t^j} \right| = |t|^3 \frac{1}{j(j-1)}$$

and this tends to zero as  $j \rightarrow \infty$  so by the ratio test the series converges for all  $t$ . Some rearranging of terms yields

$$\begin{aligned} a_{3j} &= \frac{1}{(3j)(3j-1)} \frac{1}{(3j-3)(3j-4)} \cdots \frac{1}{3 \cdot 2} a_0 \\ a_{3j+1} &= \frac{1}{(3j+1)(3j)} \frac{1}{(3j-2)(3j-3)} \cdots \frac{1}{4 \cdot 3} a_1. \end{aligned}$$

This equation is called *Airy's* equation and the two linearly independent solutions are called Airy functions.

In general we have the following result. Consider

$$x'' + p(t)x' + q(t)x = 0 \quad x(0) = a_0 \quad x'(0) = a_1$$

and suppose that

$$p(t) = \sum_{k=0}^{\infty} p_k t^k, \quad q(t) = \sum_{k=0}^{\infty} q_k t^k$$

converge in some interval  $J$  containing  $t = 0$ . Then

$$x(t) = \sum_{j=0}^{\infty} a_j t^j$$

where

$$(j+2)(j+1)a_{j+2} + \sum_{k=0}^j [p_{j-k}a_{k+1}(k+1) + q_{j-k}a_k] = 0$$

and the series converges on the whole interval  $J$ .

As a last application of regular series, we apply it to the Legendre's equation which arises in the study of certain partial-differential equations defined in a spherical domain:

$$(1-t^2)x'' - 2tx' + p(p+1)x = 0.$$

We rewrite this as

$$x'' - \frac{2t}{1-t^2}x' + \frac{p(p+1)}{1-t^2}x = 0.$$

We use the binomial expansion to expand the denominator

$$(1 - t^2)^{-1} = 1 + t^2 + t^4 + \dots + t^{2j} + \dots$$

Note that this series converges only for  $|t| < 1$ . Using the above recursion we get after some manipulation (I will leave it to you to verify this) that

$$a_{j+2} = -\frac{(p+j+1)(p-j)}{(j+2)(j+1)}a_j.$$

This series converges from the ratio test as long as  $|t| < 1$ . Furthermore note that the even terms depend only on  $a_0$  and the odd on  $a_1$ . Note that if  $p$  is an integer, the series is finite and only goes out to  $p+2$  terms. If  $p = n$  is an integer, we call the solutions the Legendre polynomials:

$$P_0 = 1, P_1 = t, P_2 = \frac{1}{2}(3t^2 - 1), P_4 = \frac{1}{2}(5t^3 - 3t).$$

## 1.1 Homework

1. Find the first few 6 terms in the series solution for

$$x'' + \cos(t)x = 0$$

2. Solve

$$x'' + tx' + x = 0$$

with  $x(0) = 1$  and  $x'(0) = 0$ . What is the interval of convergence?

3. Use series to find the first 3 nonzero terms to the solution to

$$x' = 1 + tx^2 \quad x(0) = 0$$

Note that this is a nonlinear equation.

## 2 Singularities

Many interesting physical problems involve differential equations which do not have analytic coefficients as the cases above. The simplest such equation is the Cauchy-Euler equation:

$$t^2 x'' + tp x' + qx = 0.$$

If you try to solve this with series, it will not work as you can see that the coefficient involving  $x$  is actually  $q/t^2$  which is singular at  $t = 0$ . There are several ways to solve this equation however. One is to change the independent variable letting  $t = e^s$ . (How did I know to do that? I knew since I knew what the solution would look like...) In this case (using the chain rule) we get

$$x_{ss} + (p-1)x_s + qx = 0$$

and we can easily solve this. Let  $r_1, r_2$  be the roots of the characteristic equation  $r^2 + (p-1)r + q = 0$ . If the roots are real and distinct, then  $x(s) = e^{r_1 s}$  or  $x(t) = t^{r_1}$ . If the roots are real and repeated, the other solution is  $se^{r_2 s}$  or  $x(t) = \ln(t)t^{r_2}$ . Finally, for complex roots,  $x(s) = e^{as} \cos bs$  and  $x(s) = e^{as} \sin bs$  or in terms of the original variable,  $x(t) = t^a \sin(b \ln t)$ ,  $x(t) = t^a \cos(b \ln t)$ . Note that all of these have singularities at the origin unless the roots are real and positive integers.

Lets turn to a general system

$$x'' + P(t)x' + Q(t)x = 0$$

We say that  $t = 0$  is a regular singular point if  $t^2Q(t)$  and  $tP(t)$  are analytic at  $t = 0$ . If an equation has a regular singular point, we can write it in the form

$$t^2x'' + tp(t)x' + q(t) = 0 \quad (2)$$

where  $t^2Q = q$ ,  $t^2P = p$  and  $p, q$  are now analytic functions at  $t = 0$ . Once again, we write

$$p(t) = \sum_{k=0}^{\infty} p_k t^k \quad q(t) = \sum_{k=0}^{\infty} q_k t^k.$$

Suppose that the polynomial (called the *indicial equation*)

$$f(r) = r^2 + (p_0 - 1)r + q_0$$

has real roots,  $r_1, r_2$  with  $r_2 \leq r_1$ . Then (2) has a solution of the form

$$x_1 = |t|^{r_1} \sum_{j=0}^{\infty} a_j(r_1) t^j, \quad t < 0 \quad \text{or} \quad t > 0$$

with  $a_0 \neq 0$  and

$$f(r_1 + j)a_j(r_1) = - \sum_{k=0}^{j-1} [(k + r_1)p_{j-k} + q_{j-k}]a_k(r_1).$$

The series converges in the same interval as that of  $p, q$  and if  $r_1 - r_2$  is not an integer, there is a second linearly independent solution

$$x_1 = |t|^{r_2} \sum_{j=0}^{\infty} a_j(r_2) t^j, \quad t < 0 \quad \text{or} \quad t > 0$$

where the coefficients are the same as above but  $r_2$  replacing  $r_1$ .

This looks complicated, but if you proceed stepwise, it is not so bad.

## 2.1 Bessel's equation

Bessel functions are the main reason that I have introduced this. In many linear PDEs that involve cylindrical coordinates, Bessel's differential equation arises. This is

$$t^2x'' + tx' + (t^2 - p^2)x = 0.$$

Obviously,  $t = 0$  is a regular singular point and the indicial equation is

$$r^2 - p^2 = 0.$$

The roots are  $r = \pm p$ . If  $p$  is an integer, then the procedure we have illustrated gives only one linearly independent solution. But lets just work with this solution for now. Substitute  $x_1 = t^p \sum_{j=0}^{\infty} a_j t^j$  into this obtaining

$$\begin{aligned} t^2 \sum_{j=0}^{\infty} (j+p)(j+p-1)a_j t^{j+p-2} &+ t \sum_{j=0}^{\infty} (j+p)a_j t^{j+p-1} \\ &+ t^2 \sum_{j=0}^{\infty} a_j t^j - p^2 \sum_{j=0}^{\infty} a_j t^j = 0 \end{aligned}$$

After some manipulation, this leads to

$$(1 + 2p)a_1 t + \sum_{j=2}^{\infty} [j(j+2p)a_j + a_{j-2}]x^j = 0$$

giving us the desired recursion,

$$a_1 = 0 \quad a_j = -1 \frac{a_{j-2}}{j(j+2p)}$$

which depends on the arbitrary constant  $a_0$ . If  $p$  is not an integer, then replacing  $p$  with  $-p$  yields a second solution. If  $p$  is an integer, then clearly negative values of  $p$  will cause the recursion to become singular. Otherwise, the ratio test shows that this is analytic on the whole line. We can write the series in a compact form by noting that the odd coefficients disappear ( $a_1 = 0$ ) so that

$$x_1 = a_0 x^p \left\{ 1 + \sum_{j=1}^{\infty} \frac{(-1)^j t^{2j}}{2^{2j} j! (1+p) \cdots (j+p)} \right\}.$$

Suppose that  $p = n$  is an integer and we choose  $a_0 = 1/(2^n n!)$ . Then we get the Bessel function of the first kind of integer order:

$$J_n(t) = \left(\frac{t}{2}\right)^n \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+n)!} \left(\frac{t}{2}\right)^{2j}.$$

They looked like damped trig functions.

Bessel functions have many interesting properties.

If  $r_1 = r_2$  in the indicial equation, then the second linearly independent solution is

$$x_2 = x_1 \ln |t| + |t|^{r_1} \sum_0^{\infty} c_j t^j$$

If  $r_1 - r_2$  is a positive integer, then

$$x_2 = \alpha x_1 \ln |t| + |t|^{r_2} \left(1 + \sum_1^{\infty} d_j t^j\right)$$

where  $\alpha$  is a possibly 0 constant.

As a last example, consider

$$t^2 x'' - tx = 0.$$

The indicial equation has roots 0 and 1. One solution is

$$x_1 = t \sum_{j=0}^{\infty} a_j t^j$$

and we find that

$$x_1 = \sum_{j=0}^{\infty} \frac{t^{j+1}}{(j+1)! j!}$$

Now we find the other solution. Set

$$x_2 = \alpha x_1 \ln |t| + \sum_{j=0}^{\infty} d_j t^j \quad d_0 = 1$$

We rewrite the equation in simpler form as  $Lx = tx'' - x$ . We use the fact that  $Lx_1 = 0$  to get

$$0 = Lx_2 = \alpha x_1 L \ln |t| + L \sum_{j=0}^{\infty} d_j t^j$$

and applying the required differentiations:

$$0 = \alpha \sum_{j=0}^{\infty} \frac{2j+1}{(j+1)! j!} x^j + \sum_{j=0}^{\infty} [j(j+1)d_{j+1} - d_j] x^j.$$

This leads to

$$j(j+1)d_{j+1} - d_j = -\alpha \frac{2j+1}{(j+1)!j!}.$$

This gives e.g.

$$-d_0 = -\alpha \quad 2d_2 - d_1 = \frac{-3\alpha}{2} \quad 6d_3 - d_2 = \frac{-5\alpha}{12}$$

Since  $d_0 = 1$  this determines  $\alpha$ . The rest follow. The parameter  $d_1$  is arbitrary so we set it to 0 for convenience.

## 2.2 Homework

Find series solutions to

1.

$$t^2 x'' - t/2x' + tx = 0$$

2. Laguerre's equation:

$$tx'' + (1-t)x' + px = 0$$

Hint: The answer to this is in your book (problem 10, 11.4) Show that if  $p$  is a non-negative integer, then  $x(t)$  is a polynomial. Assume that  $x(0) = 1$  and find the first 5 such polynomials ( that is  $p = 0$ ,  $p = 1$  up to  $p = 5$ . )

3.

$$t^2 x'' - tx' + tx = 0$$

(Note the indicial equation has roots that differ by an integer in this case.)