HOMEWORK 2

Due Sept 13

1. It is possible to prove existence directly using nothing more than calculus. Suppose that $f(t,x)$ is defined in $[t_0-\alpha, t_0+\alpha] \times B(\bar{a},\beta)$ and bounded with bound, M. Let $b=\min(\alpha,\beta/M)$. f has a Lipschitz constant L. Let
$$y_{n+1} = a + \int_{t_0}^{t} f(s, y_n(s))\ ds$$
with $y_0 = a$. Prove by induction that
$$|y_{n+1}(t) - y_n(t)| \le \frac{ML^n(t-t_0)^{n+1}}{(n+1)!}$$
Conclude that
$$a + \sum_{n=0}^{\infty}[y_{n+1}(t) - y_n(t)] = y(t)$$
is uniformly convergent on $[t_0-b, t_0+b]$, that is
$$y(t) = \lim_{n\to\infty} y_n(t)$$
exists uniformly. Show that $y(t)$ is a solution to
$$y(t) = a + \int_{t_0}^{t} f(s, y(s))\ ds.$$

2. Solve the integral equation:
$$x(t) = 1 + \int_0^t sx(s)\ ds.$$
Compute the first few terms for the Picard iteration and then use this to write the general n^{th} iterate. Show that the terms are same as the Taylor series expansion for your solution to the integral equation.

3. Find the maximal interval of existence to the ODE:
$$\dot{x} = x^2, \quad x(0) = a.$$

4. Suppose that $x' = -x + f(x)$ where $f(x)$ is Lipschitz continuous and $0 < f(x) < 1$ for all $x \in R$. Prove that the solution exists for all time.

5. Suppose that $x(t) > 0$ and that
$$\frac{dx}{dt} \le x^2$$
$$\int_0^\infty x(s)\ ds < \infty$$
Prove that $x(t) < \infty$ for all t. (Note: this is not obvious since the integral constraint does not eliminate singularities like $1/\sqrt{t}$.; here is a hint: write x^2 as $x(t)x(t)$ and think about how to solve $x' = a(t)x(t)$.)