FINAL EXAM 2920: Due Friday December 12th in my office before 5:00 PM

1. (20 pts) Find both Lyapunov & asymptotic stability where relevant. If the system if Hamiltonian, find the Hamiltonian. If linear stability does'nt help and the system is non-Hamiltonian, then you might try to find a Lyapunov function.

   (a) Find all equilibria and their stability and sketch the phase portrait

   $$\begin{aligned} x' &= x(1-y^2) \\ y' &= x-y \end{aligned}$$

   (b) Find all equilibria and their stability and sketch the phase-portrait:

   $$\begin{aligned} x' &= y \\ y' &= -x(1-x^2) \end{aligned}$$

   (c) Analyze the stability of $(0,0)$.

   $$\begin{aligned} x' &= -y-x^3+xy^2 \\ y' &= x-y^3 \end{aligned}$$

   (d) Find equilibria and their stability:

   $$\begin{aligned} x' &= xy \\ y' &= 1-x^2 \end{aligned}$$

   (Note this one is a bit tricky; try looking at dx/dy). Are all trajectories to this bounded? If not, find an unbounded one.

2. (5 pts) Find conditions on the parameters a, b such the the origin is an asymptotically stable critical point to the ODE:

   $$t^2x''(t) + atx'(t) + bx(t) = 0$$

3. (20 pts) (a) Prove that the solutions to $x'' + x/t = 0$ are oscillatory. (Hint consider $x'' + x/t^2 = 0$.) (b) Consider $x'' + a(t)x' + b(t)x = 0$ with initial conditions $x'(0) = 1$ and $x(0) = \alpha$. Suppose that there is a t^* such that $x_1(t^*) = x_2(t^*)$ where $x_j(t)$ satisfy $x_j(0) = \alpha_j$ and $\alpha_1 \neq \alpha_2$. Prove that for every α, $x(t^*) = x_1(t^*)$. (c) Consider the Sturm-Liouville system

   $$u'' + [\lambda p(t) - q(t)]u = 0, \quad 0 < t < 1$$

   with $p(t) > 0$ and p, q continuous on $[0,1]$. Suppose $\beta u(1) + \beta' u'(1) = 0$ and the two different sets of left-end conditions

   $$\alpha_j u(0) + \alpha_j' u'(0) = 0.$$

   Show that if $\alpha_2'/\alpha_2 < \alpha_1'/\alpha_1$, the eigenvalues of the α_2, α_2' system are smaller than the corresponding eigenvalues of the α_1, α_1' system. (d) Find the eigenvalues and eigenfunctions for for $u'' - \lambda u = 0$ with the boundary conditions, $u'(0) = 0$ and $u(1) = 0$.

4. (18 pts) (a) Let $a(t), b(t)$ be continuous and periodic with period, T. Let $\phi(t), \psi(t)$ be solutions to $x'' + a(t)x' + b(t)x = 0$ with $\psi(0) = \phi'(0) = 0$ and $\psi'(0) = \phi(0) = 1$. Show that the Floquet multipliers of the system solve $\lambda^2 - B\lambda + A = 0$ where,$B = \phi(T) + \psi'(T)$ and $A = \exp\left(-\int_0^T a(t)\ dt\right)$.
   (b) Consider the linear ODE:

$$x''(t) = q(t)x(t)$$

   where $q(t) = -1$ when t modulo $2 \in [0, 1)$ and $q(t) = b > 0$ when t modulo $2 \in [1, 2)$. (This is linear with piecewise continuous coefficients.) For what values of b is the origin Liapunov stable? Is it ever possible for the origin to be asymptotically stable? For what values of b do there exist some nonzero solutions such that $x(t) \to 0$ as $t \to \infty$? (Hints: Break the problem into two parts for $0 \le t < 1$ and $1 \le t < 2$ and match x and its derivative at the points $0, 2$ and 1. Use exercise 4 of HW 5.) (c) Consider the ODE, $x'' + a(t)x = 0$ where $a(t+1) = a(t)$ and $a(t)$ is continuous and $\int_0^1 a(t)\ dt > 0$. Prove that every solution of the form, $u(t+1) = \rho u(t)$ must have a zero in $[0, 1]$. (Try to obtain a contradiction assuming there is no zero.)

5. (10 pts) Is this statement true? No solutions to the equation $x' = f(x)$ with $f(x)$ continuous can reach an equilibrium point in finite time. If this is false, find a counterexample and add one more condition to make it true.

6. (10 pts) (a) Suppose that $V : R^n \to R$ is a continuously differentiable function and that $x' = f(x)$ is an ODE in R^n with $f(x)$ continuous. Suppose that $dV/dt \le 0$ along trajectories and that this orbital derivative vanishes only at equilibria. Prove there are no periodic solutions. (b) Suppose that $F : R^3 \to R^3$ is a continuously differentiable function such that $\nabla \times F = 0$ (That is, it has zero curl.) Prove that the vector field, $X' = F(X)$ has no periodic orbits.

7. (10 pts) (a) Let $F, G : R^2 \to R^2$ be continuous planar functions such that $F(X) \cdot G(X) = 0$ for all X and such that the vector field, $X' = F(X)$ contains a clsed orbit. Show the vector field, $X' = G(X)$ has an equitlibrium point. (b) Consider the planar ODE $x' = f(x, y)$, $y' = g(x, y)$. Suppose that the origin is the only equilibrium and that it is asymptoticaly stable and that for $x^2 + y^2$ sufficiently large, $xf(x, y) + yg(x, y) > 0$. Prove that there exists at least one unstable limit cycle.

8. (6 pts) Describe the qualitative types of solutions possible to the linear constant-coefficient ODE, $x' = Ax$ given that (a) $A = A^T$ or (b) $A = -A^T$. (This is really an exercise in linear algebra.)