

FINAL EXAM 2920: Due Friday December 12th in my office before 5:00 PM

1. (20 pts) Find both Lyapunov & asymptotic stability where relevant. If the system is Hamiltonian, find the Hamiltonian. If linear stability doesn't help and the system is non-Hamiltonian, then you might try to find a Lyapunov function.

- (a) Find all equilibria and their stability and sketch the phase portrait

$$\begin{aligned}x' &= x(1 - y^2) \\y' &= x - y\end{aligned}$$

- (b) Find all equilibria and their stability and sketch the phase-portrait:

$$\begin{aligned}x' &= y \\y' &= -x(1 - x^2)\end{aligned}$$

- (c) Analyze the stability of  $(0,0)$ .

$$\begin{aligned}x' &= -y - x^3 + xy^2 \\y' &= x - y^3\end{aligned}$$

- (d) Find equilibria and their stability:

$$\begin{aligned}x' &= xy \\y' &= 1 - x^2\end{aligned}$$

(Note this one is a bit tricky; try looking at  $dx/dy$ ). Are all trajectories to this bounded? If not, find an unbounded one.

2. (5 pts) Find conditions on the parameters  $a, b$  such that the origin is an asymptotically stable critical point to the ODE:

$$t^2 x''(t) + atx'(t) + bx(t) = 0$$

3. (20 pts) (a) Prove that the solutions to  $x'' + x/t = 0$  are oscillatory. (Hint consider  $x'' + x/t^2 = 0$ .) (b) Consider  $x'' + a(t)x' + b(t)x = 0$  with initial conditions  $x'(0) = 1$  and  $x(0) = \alpha$ . Suppose that there is a  $t^*$  such that  $x_1(t^*) = x_2(t^*)$  where  $x_j(t)$  satisfy  $x_j(0) = \alpha_j$  and  $\alpha_1 \neq \alpha_2$ . Prove that for every  $\alpha$ ,  $x(t^*) = x_1(t^*)$ . (c) Consider the Sturm-Liouville system

$$u'' + [\lambda p(t) - q(t)]u = 0, \quad 0 < t < 1$$

with  $p(t) > 0$  and  $p, q$  continuous on  $[0, 1]$ . Suppose  $\beta u(1) + \beta' u'(1) = 0$  and the two different sets of left-end conditions

$$\alpha_j u(0) + \alpha'_j u'(0) = 0.$$

Show that if  $\alpha'_2/\alpha_2 < \alpha'_1/\alpha_1$ , the eigenvalues of the  $\alpha_2, \alpha'_2$  system are smaller than the corresponding eigenvalues of the  $\alpha_1, \alpha'_1$  system. (d) Find the eigenvalues and eigenfunctions for  $u'' - \lambda u = 0$  with the boundary conditions,  $u'(0) = 0$  and  $u(1) = 0$ .

4. (18 pts) (a) Let  $a(t), b(t)$  be continuous and periodic with period,  $T$ . Let  $\phi(t), \psi(t)$  be solutions to  $x'' + a(t)x' + b(t)x = 0$  with  $\psi(0) = \phi'(0) = 0$  and  $\psi'(0) = \phi(0) = 1$ . Show that the Floquet multipliers of the system solve  $\lambda^2 - B\lambda + A = 0$  where,  $B = \phi(T) + \psi'(T)$  and  $A = \exp\left(-\int_0^T a(t) dt\right)$ .  
 (b) Consider the linear ODE:

$$x''(t) = q(t)x(t)$$

where  $q(t) = -1$  when  $t$  modulo 2  $\in [0, 1)$  and  $q(t) = b > 0$  when  $t$  modulo 2  $\in [1, 2)$ . (This is linear with piecewise continuous coefficients.) For what values of  $b$  is the origin Liapunov stable? Is it ever possible for the origin to be asymptotically stable? For what values of  $b$  do there exist some nonzero solutions such that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ ? (Hints: Break the problem into two parts for  $0 \leq t < 1$  and  $1 \leq t < 2$  and match  $x$  and its derivative at the points 0, 2 and 1. Use exercise 4 of HW 5.) (c) Consider the ODE,  $x'' + a(t)x = 0$  where  $a(t+1) = a(t)$  and  $a(t)$  is continuous and  $\int_0^1 a(t) dt > 0$ . Prove that every solution of the form,  $u(t+1) = \rho u(t)$  must have a zero in  $[0, 1]$ . (Try to obtain a contradiction assuming there is no zero.)

5. (10 pts) Is this statement true? No solutions to the equation  $x' = f(x)$  with  $f(x)$  continuous can reach an equilibrium point in finite time. If this is false, find a counterexample and add one more condition to make it true.
6. (10 pts) (a) Suppose that  $V : R^n \rightarrow R$  is a continuously differentiable function and that  $x' = f(x)$  is an ODE in  $R^n$  with  $f(x)$  continuous. Suppose that  $dV/dt \leq 0$  along trajectories and that this orbital derivative vanishes only at equilibria. Prove there are no periodic solutions. (b) Suppose that  $F : R^3 \rightarrow R^3$  is a continuously differentiable function such that  $\nabla \times F = 0$  (That is, it has zero curl.) Prove that the vector field,  $X' = F(X)$  has no periodic orbits.
7. (10 pts) (a) Let  $F, G : R^2 \rightarrow R^2$  be continuous planar functions such that  $F(X) \cdot G(X) = 0$  for all  $X$  and such that the vector field,  $X' = F(X)$  contains a closed orbit. Show the vector field,  $X' = G(X)$  has an equilibrium point. (b) Consider the planar ODE  $x' = f(x, y)$ ,  $y' = g(x, y)$ . Suppose that the origin is the only equilibrium and that it is asymptotically stable and that for  $x^2 + y^2$  sufficiently large,  $xf(x, y) + yg(x, y) > 0$ . Prove that there exists at least one unstable limit cycle.
8. (6 pts) Describe the qualitative types of solutions possible to the linear constant-coefficient ODE,  $x' = Ax$  given that (a)  $A = A^T$  or (b)  $A = -A^T$ . (This is really an exercise in linear algebra.)