1. Consider the competition between two species:

\[ x' = x(1 - x - ay) \]
\[ y = y(1 - y - bx) \]

where \( a, b \) are positive. (a) Show that \((0, 0), (1, 0), (0, 1)\) are all equilibria and interpret what they mean. (b) Determine the stability as a function of \((a, b)\) (c) There is a fourth equilibrium \([x_4, y_4] = [(1 - a)/(1 - ab), (1 - b)/(1 - ab)]\) as long as \( ab \neq 1 \). For what choices of \( a, b \) are both \( x_4, y_4 \) positive? The linearization matrix at this equilibrium point is

\[ J = \begin{pmatrix}
-x_4 & -ax_4 \\
-by_4 & -y_4
\end{pmatrix} \]

Assume \( a, b \) are such that \( x_4, y_4 \) are both positive. Show that the equilibrium is stable if \( ab < 1 \). (Hint what is the trace? what is the determinant?) Note you do not have to substitute the values of \( x_4, y_4 \) into the matrix to answer this!! (d) Sketch the phase plane for \( a = b = .5 \) and for \( a = b = 2 \) including the nullclines. Make sure you draw a few trajectories and show which equilibria are stable. You only need to draw the phase plane for \( x, y \) non-negative.

**Answer. 19 pts (a) 3 pts.** Clearly \( x = 0 \) is an equilibrium for the \( x \) equation and in this case, the \( y \) equation is \( y(1 - y) \) which has \( y = 0, 1 \) as zeros. Similarly \( y = 0 \) solves the \( y \) equation and then \( x = 0, 1, 0, 0 \) means all populations are dead, \((1, 0)\) means only \( x \) survives, and \((0, 1)\) means only \( y \) survives. (b)\((6 pts; 2 each)\) the Jacobian is

\[ J = \begin{pmatrix}
1 - 2x - ay & -ax \\
-by & 1 - 2y - bx
\end{pmatrix} \]

From this, we get

\[ J_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

so \((0, 0)\) is unstable;

\[ J_{(1,0)} = \begin{pmatrix} -1 & -a \\ 0 & 1 - b \end{pmatrix} \]

So \((1, 0)\) is stable if \( b > 1 \) and unstable if \( b < 1 \).

\[ J_{(0,1)} = \begin{pmatrix} 1 - a & 0 \\ -b & -1 \end{pmatrix} \]

So \((0, 1)\) is stable if \( a > 1 \) and unstable of \( a < 1 \). (c)\((4 pts)\) This equilibrium is positive if \( a < 1, b < 1 \) or \( a > 1, b > 1 \). The trace is \(-(x_4 + y_4) < 0 \) since both positive. The determinant is \( x_4y_4(1 - ab) \). This is positive if \( ab < 1 \). We conclude \((x_4, y_4)\) is stable if \( a < 1, b < 1 \). (d)\((6 pts, 3 each)\) Need to include trajectories, nullclines. Optional - labeling the fixed points. (see figures below)
2. For the ecological networks shown in the figure, (a) write the matrix of signs, (b) which are guaranteed to be stable just based on the signs of the arrows. Please explain your reasoning. (c) What sort of system could they describe?

*Answer. 18 pts* (a) 8 points, 2 each

\[ Q_a = \begin{pmatrix} 0 & - & 0 & 0 \\ + & 0 & 0 & 0 \\ 0 & + & 0 & - \\ 0 & 0 & - & - \end{pmatrix} \]

\[ Q_b = \begin{pmatrix} 0 & 0 & 0 & 0 & - \\ + & 0 & + & 0 & 0 \\ 0 & - & 0 & + & 0 \\ 0 & 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 & - \end{pmatrix} \]

\[ Q_c = \begin{pmatrix} 0 & 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & 0 \\ + & - & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & - & 0 & - \\ 0 & 0 & 0 & + & - & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & - \end{pmatrix} \]

\[ Q_d = \begin{pmatrix} - & - & 0 & - \\ + & 0 & - & 0 \\ 0 & + & 0 & - \\ 0 & 0 & + & 0 \end{pmatrix} \]

(b) (8 points, 2 each) (a) is not guaranteed to be stable because \( q_{43}q_{34} > 0 \). (b) is guaranteed to be stable because \( q_{ii} \leq 0 \) and \( q_{55} < 0 \); \( q_{ij}q_{ji} \leq 0 \); there are no cycles bigger than 2; the graph is connected. (c) is guaranteed to be stable since \( q_{ii} \leq 0 \) and \( q_{55} < 0 \), \( q_{77} < 0 \); \( q_{ij}q_{ji} \leq 0 \); there are no cycles bigger than 2; the graph is connected. (d) is not guaranteed to be stable since there is a cycle 1234. (c) They can write anything here. It is just making up a story. 2 points

3. Consider the SIS (susceptible -infected-susceptible) disease model:

\[ S' = \delta(N - S) - \beta IS + \gamma I \]

\[ I' = \beta IS - (\delta + \nu + \gamma)I \]

with natural death rate \( \delta \), recovery, \( \gamma \), and death from the disease, \( \nu \). What is \( R_0 \) for this disease. Find the equilibria and their stability. When is the disease endemic?

*Answer. 12 pts* From the \( I \) equation, \( I' = (\beta S - (\gamma + \delta + \nu))I \), and at the beginning \( S = N \), so to get \( I' > 0 \) we need \( \beta N > (\gamma + \delta + \nu) \) or
\[ R_0 = \frac{(\beta N)}{(\gamma + \delta + \nu)} > 1 \quad \text{(2 pts)} \] The equilibria are \((N, 0)\) \((1 \text{ pt})\) (no disease) and

\[
S_1 = \frac{(\gamma + \delta + \nu)}{\beta} \\
I_1 = \frac{\delta(N - S_1)}{(\nu + \delta)} = \frac{\delta N}{\nu + \delta} \left(1 - \frac{1}{R_0}\right)
\]

(One could substitute \(S_1\) into this, but we don’t need to, and if you factor out \(N\), \(R_0\) shows up.) \((3 \text{ pts})\) The Jacobian is

\[
J = \begin{pmatrix}
-\delta - \beta I & \gamma - \beta S \\
\beta I & \beta S - (\delta + \nu + \gamma)
\end{pmatrix}
\]

At \((N, 0)\)

\[
J_0 = \begin{pmatrix}
-\delta & \gamma - \beta S \\
0 & \beta N - (\delta + \nu + \gamma)
\end{pmatrix}
\]

This is stable if \(\beta N - (\delta + \nu + \gamma) < 0\), or \(R_0 < 1\). \((2 \text{ pts})\) At the other equilibrium:

\[
J_1 = \begin{pmatrix}
-\delta - \beta I_1 & -(\nu + \delta) \\
\beta I_1 & 0
\end{pmatrix}
\]

We assume \(I_1 > 0\) (since otherwise, the equilibrium makes no sense). Then the trace is clearly negative. The determinant is \((\nu + \delta)\beta I_1\) which is positive when \(I_1 > 0\), so this is stable when \(I_1 > 0\). \((3 \text{ pts})\) The disease is endemic if \(R_0 > 1\) or \(\beta N > (\gamma + \delta + \nu)\). \((1 \text{ pt})\)
Figure 1: Community networks for problem 2
Figure 2: Answer to 1d