37 points total

NOTE: It is OK if they do not show their work, say Maple or Mathematica. However, then there is no basis for partial credit. So you can feel free to give partial credit if they include their algebra since you can then tell if they just mistyped.

1. 12 points total (a) 4 points Determine the number of whales that maximizes the growth new whales born. I let $x$ be blue and $y$ be fin. I maximized $\frac{dx}{dt} + \frac{dy}{dt}$ a function of 2 variables to get

$$x = \frac{r_2K_1((a_1 + a_2)K_2 - 2r_1)/(K_1(a_1 + a_2)^2K_2 - 4r_1r_2)}{K_1((a_1 + a_2)K_1 - 2r_2)K_2/(K_1(a_1 + a_2)^2K_2 - 4r_1r_2)}$$

$$y = \frac{r_1((a_1 + a_2)K_1 - 2r_2)K_2/(K_1(a_1 + a_2)^2K_2 - 4r_1r_2)}{K_1((a_1 + a_2)K_1 - 2r_2)K_2/(K_1(a_1 + a_2)^2K_2 - 4r_1r_2)}$$

For the parameters in the book, I get $x = 69103$ blue whales and $y = 196544$ fin whales While it is not necessary to so all the steps in the five step method, it can sometimes be helpful, so

• Step 1: Ask the question. What numbers of whales maximize the growth rate
• Step 2: Modeling approach; unconstrained optimization
• Step 3: Formulate the model; this is done already, just maximize the rates as above
• Step 4: Solve it. Above is solution
• Step 5: Answer question. See above

(b) 4 points Find sensitivity of the populations to $r_1, r_2$. Using Maple, I get: $S(x,r_1) = .085, S(y,r_1) = -.0015, S(x,r_2) = -.0015, S(y,r_2) = .0176$. (c) 2 points Again, using Maple, I get: $S(x,K_1) = 1, S(y,K_1) = -.017, S(x,K_2) = -.0854, S(y,K_2) = 1.001$ (d) Assuming $a_1 = a_2 = a$, is it ever optimal to let one of the species go 2 points extinct. So, If you plot the optimal values as a function of $a$, you find that the optimal blue whales go way down and when $a = 1210^{-8}$ (12 fold higher), then the optimal blue whale population is 0. So the answer is yes!

2. Problem 3.14 points total (a) 6 points

• Step 1. Given we want to harvest the whales, we want to maximize the profit.
• Step 2. We will use unconstrained optimization
• Step 3. here is the main part. Formulate the problem. Let $h_1, h_2$ be the harvest rates of the whale in whale per year. The profit, $P = 12000h_1 + 6000h_2$ and we want to maintain the populations $x, y$ at constant values. So with a constant harvest rate, $\frac{dx}{dt} = f(x, y) - h_1$ and $\frac{dy}{dt} = g(x, y) - h_2$ where $f, g$ are the natural growth rates of the whales without harvesting. We assume that the populations are
kept constant, so this means that \( dx/dt = dy/dt = 0 \). To attain this, we need, \( h_1 = f(x, y) \) and \( h_2 = g(x, y) \), so we must maximize:

\[
P(x, y) = 12000f(x, y) + 6000g(x, y)
\]

- Step 4. Solve. Again with Maple, I get a general formula for the maximum and with the parameters of the book, \( x = 70619, y = 194703 \) and a profit of $67,914,605.
- Step 5. Answer is above

(b) **4 points** Sensitivity of \( x, y \) to \( r_1, r_2 \) are

\[
[S(x, r_1), S(y, r_1), S(x, r_2), S(y, r_2)] = [0.062, -0.00169, -0.00173, 0.0272]
\]

(c) **2 points** Sensitivity of the profit to \( r_1, r_2 \) respectively is 0.33, 0.706

(d) **2 points** As before, I find that somewhere around \( a = 1710^{-8} \), the blue whales are driven to extinction and only the fin whales are around.

3. **11 points total.** Newspaper problem (a) **6 points**

- Step 1. Find advert rate and subscription rate to maximize profit
- Will use unconstrained optimization
- Formulate. This is all the work. The approach is like the rebate problem. Let \( x \) denote the increase in price in 10 cent increments from base of $1.50 and let \( y \) be increase in advertisement price in $100 increments from base of $250. Each such subscription increment results in a loss of 5000 subscribers from the base of 80000 and the advertising increase in a loss of 50 pages from a base of 350. Finally, for every loss of 50 pages you lose 1000 subscribers. Thus let \( n \) be number of subscribers, \( p \) be pages of advertising, let \( c \) be the cost of a subscription, \( k \) the cost per page of an ad, \( a \) be rate of subscribers loss (here is is 5000) and \( b \) be the rate of loss of pages of advertising (here 50) in anticipation of the second part of the question. Let the profit \( R \) be the total revenue from ads and subscriptions. Since 50 pages are lost for each $100 increase in advertising and we lose 1000 subscribers for such a loss, we have:

\[
\begin{align*}
n &= 80000 - ax - 1000y \\
p &= 350 - by \\
c &= 1.50 + 0.1x \\
k &= 250 + 100y \\
R &= nc + pk
\end{align*}
\]

Our goal is to maximize \( R \).

- Step 4: Answer the problem. Okay, so using Maple, I find that \( x = 0.29 \) and \( y = 2.09 \) which means that we should increase the subscription rate by about 3 cents a week to $1.53 and the advertising rate by $209 to $459
(b) **2 points** $x$ is really sensitive to your choices of $a$, $S(x, a) = -26.86$ and $S(y, a) = .037$  

(c) **2 points** $S(x, b) = 1.15$, $S(y, b) = -1.59$  

(d) **1 point** - they could write anything here since it was sort of ambiguous to me. Since advertisers would have to pay $500/page and the price we found is $459$, it is still a bargain for the advertisers, so I guess you could even charge more without losing so many pages and this may throw some doubt into how many pages would be lost due to a price increase.