

Homework # 4

1. We have seen that the equation:

$$\frac{dN}{dt} = rN(1 - N/K), \quad N(0) = N_0$$

has a solution

$$N(t) = \frac{K}{1 + [(K - N_0)/N_0]e^{-rt}}$$

Use this to conclude that

$$N(t) = \frac{K}{1 + \exp\{-rt + \ln[(K - N_0)/N_0]\}}$$

Rearrange this to show

$$\frac{K - N}{N} = \exp\left(-rt + \ln \frac{K - N_0}{N_0}\right)$$

and thus that

$$\ln \frac{K - N}{N} = -rt + \ln \frac{K - N_0}{N_0}.$$

Note that this is a linear equation in $\ln[(K - N)/N]$ with slope $-r$. Gause (see picture on the next page) grew two kinds of yeast separately and observed the following data:

Age (hr)	<i>Sacch</i>	<i>Schizo</i>
6	0.37	–
16	8.87	1.00
24	10.66	–
29	12.50	1.7
40	13.27	–
48	12.87	2.73
53	12.70	–
72	–	4.87
93	–	5.67
141	–	5.83

- (a) Use the maximum population levels to estimate K_1, K_2 , the carrying capacities for the two species.
- (b) Plot $(K_1 - N_1)/N_1$ on a logscale and use this to determine r_1, r_2 . (I used Excel and did a regression analysis, but I think eyeballing this small amount of data works just as well.) Note that Gause estimated $(K_1, K_2) = (13, 5.8)$ and $(r_1, r_2) = (.21827, .06069)$; is this close to your estimate?

2. Estimating the species competition rates. Now that we have the carrying capacity and growth rates of the yeast individually, we want to estimate the competition in the following equations:

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - \beta_{12} N_2\right), \quad \frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2} - \beta_{21} N_1\right)$$

- (a) Use this equation to show that

$$\beta_{12} = \frac{r_1 N_1 K_1 - K_1 (dN_1/dt) - r_1 N_1^2}{K_1 r_1 N_1 N_2}$$

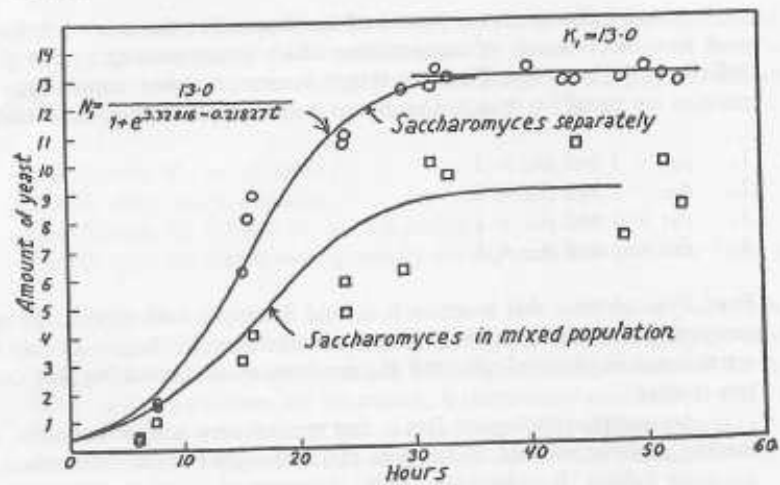
$$\beta_{21} = \frac{r_2 N_2 K_2 - K_2 (dN_2/dt) - r_2 N_2^2}{K_2 r_2 N_1 N_2}$$

- (b) Suggest how to use this equation to estimate these parameters for the figure on the next page. (Hint, consider the steady state, $dN_j/dt = 0$ and use the final values of the yeast when grown together.)
- (c) Gause estimated $(\beta_{12}, \beta_{21}) = (.24, .0756)$ using these parameters, do we expect the species to coexist?
3. Write a reasonable set of differential equations for the interaction of rabbits, foxes, and weasels. Assume that the foxes eat mostly rabbits but occasionally eat weasels and that without these two sources of food, the foxes will die. Assume that the weasels can survive without rabbits and grow to a limited carrying capacity, but they will eat the occasional rabbit. Rabbits have a large but finite carrying capacity to which they will grow in absence of the predators.
4. Write equations for a three-way competition between X, Y, Z such that when $Z = 0$, then X beats Y ; when $Y = 0$ then Z beats X ; and when $X = 0$, then Y beats Z . Now suppose that they all race together. Simulate this and see what happens.
5. Consider the following predator prey model:

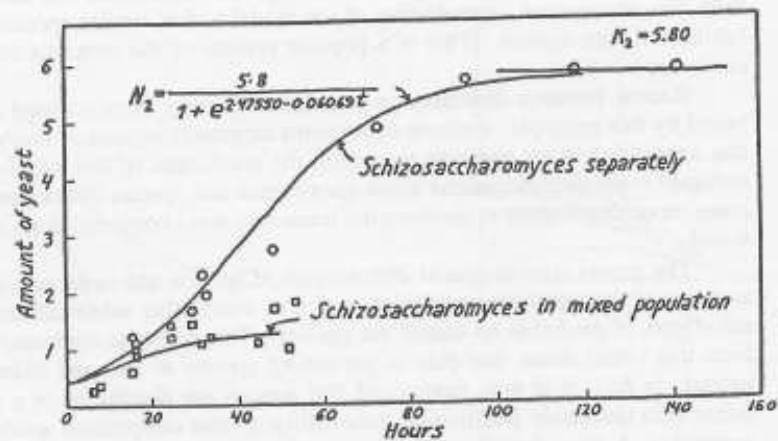
$$\frac{dx}{dt} = x[(x + b)(1 - x) - y], \quad \frac{dy}{dt} = y(-a + x).$$

Assume all parameters are positive.

- (a) Which is the predator and which is the prey?
- (b) Find the fixed points and their stability.
- (c) Draw the nullclines and the phase portrait for $b = 0.25$ and $a = .5$. Do the same for $b = 0.25, a = 0.3$. Does this system have limit cycle solutions?
- (d) Can the predator die out and just leave the prey?
- (e) For what parameters do you expect to see limit cycle solutions?



(a)



(b)

Figure 6.9 Growth of (a) *Saccharomyces cerevisiae* and (b) *Schizosaccharomyces kephir* in original experiments by Gause (1932). The organisms were grown separately (open circles) as well as in a mixed culture

containing both (open rectangles). From Gause, G. F. (1932), *Experimental studies on the struggle for existence. I. Mixed population of two species of yeast*. J. Exp. Biol., 9, Figures 2 and 3.

Gause: Empirical Tests of the Species-Competition Model

In his book, *The Struggle for Existence*, Gause (1934) describes a series of laboratory experiments in which two yeast species, *Saccharomyces cerevisiae* and *Schizosaccharomyces kephir* were grown separately and then paired in a mixed population (Figure