HW 10 answers 47 points

1. 11 points (2 in chapter 7) (a) 5 points Let $S_n = X_1 + \ldots + X_n$ and we have 3 bad diodes so $S_n = 3$ and $n = 1000$. For a binomial $\mu = q$ and $\sigma^2 = q(1 - q)$ is the variance. The Central Limit Theorem says that with 95% prob

$$-2\sigma/\sqrt{n} \leq S_n/n - \mu \leq 2\sigma/\sqrt{n}$$

so this means

$$|0.003 - q| \leq 2\sqrt{q(1 - q)/1000}$$

Or

$$f(q) = 1000(0.003 - q)^2 - 4q(1 - q) \leq 0$$

Finding the roots of this with Maple gives $0.001 \leq q \leq 0.009$ so that the best we can say is the true proportion is between 0.001 and 0.009. (b) 2 points repeat this with 30 bad diodes out of 10000, and we get $S_n/n$ is still 0.003 and we get $0.0021 \leq q \leq 0.0043$. (c) 3 points Suppose $n$ diodes are tested. We are 95% sure that the true proportion, $q$, satisfies

$$n(0.003 - q)^2 - 4q(1 - q) \leq 0$$

We want to find an $n$ so that the upper root is no more than 10% bigger, i.e. .0033 or the lower root is no more than 10% smaller, i.e. .0027. Plotting the roots over a range of $n$, we see that $n$ must be 120000 or so for the lower root and 150000 for the upper. (d) 1 point To get within 1% we need to test about 14,000,000

2. 10 points Prob 6abc Ch 7. Poisson distribution. (a) 5 points expectation

$$\sum_{n=0}^{\infty} ne^{-\lambda t}(\lambda t)^n/n! = \lambda te^{-\lambda t} \sum_{n=1}^{\infty} (\lambda t)^{n-1}/(n-1)! = \lambda t$$

where we use the Taylor series for the exponential. Variance is $E(n^2) - E(n)^2$ so lets find $E(n^2)$

$$\sum_{n=0}^{\infty} n^2 e^{-\lambda t}(\lambda t)^n/n! = \sum_{n=1}^{\infty} n(n-1) + ne^{-\lambda t}(\lambda t)^n/n! = e^{-\lambda t}(\lambda t)^2 \sum_{n=2}^{\infty} (\lambda t)^{n-2}/(n-2)! + \lambda t

= (\lambda t)^2 + (\lambda t)$$

where we used the first part and Taylor series again. Thus subtracting, we get that the variance is $\lambda t$ (b) 3 points The probability that the number of calls is within 18 of 171 is the probability that $N_t$ is between 153 and 189 is

$$\sum_{n=153}^{189} e^{-171}171^n/n! = .843$$
so 84.3% chance that we would be within 18 calls of 171. (c) 2 points To get within the 95% probability, we just use Maple to evaluate this sum over $171 \pm k$ for various values of $k$. For $k = 25$, we get 94.9% which is what we found with the normal approximation.

3. 9 points total (a) 2 points Chance of winning is $1/1000 = 0.001$, so chance of losing is $.999$. Chance of losing all 52 weeks is $0.999^{52} = 0.9493$ so chance of winning at least once is $1 - 0.9493 = 0.0507$ (b) 2 points If you buy $n$ tickets a week, the chance of winning is $1 - (1 - n/1000)^{52}$. For example, with 9 tickets a week, you have a 37.5% chance to win. If you buy tickets $n$ days a week $n \leq 7$, then chance of winning is $1 - (1 - 0.999)^{52n}$.

(c) 5 points Let $X_n$ be the states income for the $n$th ticket sold. $X_n = 1$ with probability 0.999 and $X_n = -499$ with probability 0.001 (since they have to give you 500 but you still bought the ticket so 499). Thus

\[
\mu = (-499)(0.001) + (1)(0.999) = 0.500
\]
\[
\sigma^2 = (-499 - 0.5)^2(0.001) + (1 - 0.5)^2(0.999) = 249.75
\]
\[
\sigma = \sqrt{249.75} = 15.8035
\]

By the central limit theorem with 95% confidence

\[
N\mu - 2\sigma\sqrt{N} \leq X_1 + \cdots + X_N \leq N\mu + 2\sigma\sqrt{N}
\]

so with these numbers and $N = 10^6$ tickets the states take or profit is $500000 \pm 31607$ They make out like gangsters. (d) Ignore it

4. 8 points 12 in chapt 7. (a) 3 points A randomly selected channel is busy with probability $(100/4096) = 0.0244$. A typical busy channel is transmitting about 50% of the time, so the probability of detecting a signal at any given time on a randomly selected channel is 0.0122. Then we expect to scan $1/0.0122 = 81.92$ channels in order to find one transmission. This takes 8.192 seconds, and then it takes an additional 5 seconds to get a location fix. Then we expect one detection every 13.192 seconds or 0.0758 detections per second. The expected time to detect a given individual busy channel is 1319.2 seconds or about 22 minutes. (since there are 100 channels) (b) 5 points There are 25 channels that we scan 10 times per sequence, so we will visit $25 \times 10 = 250$ busy channels for every $(4096 - 25) + 10 \times 25 = 4321$ scans, so that a randomly selected channel is busy with probability $(325/4321) = 0.0752$. The probability of detecting a signal at any given time on a randomly selected channel is 0.0376 (since active half the time) Then we expect to scan $1/0.0376 = 26.59$ channels in order to find one transmission. This takes 2.659 seconds, and then it takes an additional 5 seconds to get a location fix. Then we expect one detection every 7.659 seconds or 0.13 detections per second. For one of the 75 busy channels scanned once per cycle, the expected time to detect is $325 \times 7.659 = 2490$ seconds or about 41 minutes (since it is only one of 325 possibilities) For one of the 25 priority channels, the expected time
to detect is $32.5 \times 7.659 = 249$ seconds or about 4.2 minutes (since one of
325/10 possibilities).

5. 9 points Prob 14 chapt 7. (a) 3 points

\[
\Pr\{X > i\} = \sum_{j=i+1}^{\infty} \Pr\{X = j\} \\
= \sum_{j=i+1}^{\infty} p(1-p)^{j-1} \\
= p(1-p)^i \sum_{j=i+1}^{\infty} p(1-p)^{j-(i+1)} \\
= p(1-p)^i \sum_{k=0}^{\infty} p(1-p)^k \\
= p(1-p)^i \frac{1}{1 - (1-p)} \\
= (1-p)^i
\]

(b) 3 points For any nonnegative integers, $i, j$ we have

\[
\Pr\{X > (i+j)|X > j\} = \frac{\Pr\{X > (i+j)\}}{\Pr\{X > j\}} \\
= \frac{(1-p)^{i+j}}{(1-p)^j} = \Pr\{X > i\}
\]

(c) 2 points Differentiate $1 + x + x^2 + \cdots = 1/(1-x)$ wrt $x$ to get

\[
0 + 1 + 2x + 3x^2 + \cdots = 1/(1-x)^2 \text{ So}
\]

\[
EX = \sum_{i=1}^{\infty} i p (1-p)^{i-1} \\
= p \sum_{i=1}^{\infty} i (1-p)^{i-1} \\
= \frac{p}{(1-(1-p))^2} = 1/p.
\]

(d) 1 point if $Y$ is exponential with rate $\lambda = 1/10$ per minute, the prob
of no arrivals in first 5 minutes is $e^{-5/10} = 0.60663$. If $X$ is geometric with
$p = 0.1$, the prob of no arrivals in first 5 minutes is prob that one comes
after 5, $(1-p)^5 = 0.59045$ which is close to the exponential.