First order linear equations

Consider the following equation:

$$\frac{dx}{dt} = a(t)x + b(t)$$

with initial conditions $x(t_0) = c$. We would like to find the general solution. First, we consider the homogeneous problem:

$$\frac{dx}{dt} = a(t)x.$$

This can be solved by separation of variables:

$$\frac{dx}{x} = a(t)dt$$

which is integrated leading to

$$\ln x = K + \int a(s)ds$$

so we get

$$x_H(t) = Ce^{A(t)}$$

where $A(t) = \int a(t)dt$. The unknown constant is found by setting $t = t_0$ and using the initial data. For example:

$$\frac{dx}{dt} = x/(1+t) \qquad x(0) = 1$$

we get $a(t) = 1/(1+t)$ and $A(t) = \ln(1+t)$ so that

$$x_H(t) = Ce^{\ln(1+t)} = C(1+t)$$

so that $x_H(t) = 1 + t$.

Now, how do we solve the general system. We introduce a useful trick called the integrating factor. We can rewrite the equation as

$$\frac{dx}{dt} - a(t)x = b(t).$$

We call $F(t) \equiv e^{A(t)}$ the integrating factor. Note that

$$\frac{dF}{dt} - a(t)F(t) = 0.$$

Let $x(t) = y(t)F(t)$. Let's differentiate this:

$$\begin{aligned} \frac{dx}{dt} - a(t)x(t) &= \frac{dy}{dt}F(t) + y(t)\frac{dF}{dt} - a(t)y(t)F(t) \\ &= F(t)\frac{dy}{dt} \end{aligned}$$

since $dF/dt = a(t)F = 0$. Thus, we see that since $dx/dt - a(t)x = b(t)$, we have

$$F(t)\frac{dy}{dt} = b(t)$$

so that

$$\frac{dy}{dt} = \frac{b(t)}{F(t)} = b(t)e^{-A(t)}$$

This can be integrated:

$$y(t) = \int e^{-A(s)}b(s)ds + K$$

and since $x(t) = y(t)F(t)$, we get

$$x(t) = Ke^{A(t)} + e^{A(t)} \int^t e^{-A(s)}b(s)ds$$

is the general solution.

Example.

Solve

$$x' = x/(1+t) + 2t \qquad x(0) = 3.$$

$a(t) = 1/(1+t)$. Thus $A(t) = \ln(1+t)$ and $F(t) = e^{A(t)} = 1 + t$. So

$$x(t) = K(1+t) + (1+t)\int_0^t \frac{2s}{1+s}ds = K(1+t) + 2t - 2\ln(1+t)$$

The initial data yields, $K = 3$ so that

$$x(t) = 3 + 5t - 2\ln(1+t).$$

Homework Solve the following:

1. $x' = -x/t + t$ with $x(1) = 0$
2. $x' = -4x + \sin(3t)$ with $x(0) = 0$.
3. $x' = -t^2x + t$ with $x(0) = 1$.
4. $x' = -2x/t + \sin(t)$ with $x(\pi) = 1/\pi$

For problems 2 and 4, describe the behavior as $t \to \infty$.

Note the third one should convince you of the worthlessness of exact methods of solution in general.