Homework

1. Find the general solutions to each of the following systems (don't forget to include the solution to the homogeneous problem in your answer)

   (a)

$$X' = \begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix} X + \begin{pmatrix} e^{-t} \\ e^{-2t} \end{pmatrix}$$

   (b)

$$X' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} X + \begin{pmatrix} \sec 2t \\ 1 \end{pmatrix}$$

   (c)

$$X' = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} X + \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}$$

   (d) Solve the above problem for $X(0) = [1, -1]^T$.

2. Find the general solution to:

$$X' = \begin{pmatrix} 4 & 6 \\ -1 & -1 \end{pmatrix} X + \begin{pmatrix} -3e^{-t} \\ 2e^{-2t} \end{pmatrix}.$$

   For what initial condition $X(0)$ do solutions decay to zero as $t \to \infty$?

3. Find the general solutions to the problems $P(D)x = f(t)$ where

   (a) $P(u) = (u + 2)(u - 1)$ and $f(t) = e^t$

   (b) $P(u) = u^2 + 2u + 10$ and $f(t) = e^{-t}$

   (c) $P(u) = u^2 + 1$ and $f(t) = \sin 2t$, $f(t) = \sin t$, $f(t) = \sin \sqrt{2}t$. For each of these, determine which solutions are periodic.

   (d) $P(u) = (u + 3)^3$ and $f(t) = e^{-3t}$. For this problem, write down the solution which satisfies, $x(0) = 0$ and $x'(0) = 0$.

4. For each of these equations of the form $P(D)x = 0$, write them as a system of equations, determine the companion matrix, C and find the matrix solution $W(t)$ whose columns are the linear independent solutions to $Y' = CY$ (a) $P(u) = u(u^2 - 1)$; (b) $P(u) = u^3(u - 1)$; (c)$P(u) = (u + 1)^2 u$.

5. Suppose that A is a two-dimensional matrix with real negative eigenvalues. Suppose that b is a constant 2-dimensional vector. Consider $X' = AX + b$. Find the equilibrium point, X_{eq} of this system and prove that it is unique. Prove that all solutions to $X' = AX + b$ converge to X_{eq} as $t \to \infty$.