1 Mechanical systems with one degree of freedom

Many simple physical systems can be described by a point mass operating under a force field without friction. Examples are the simple harmonic oscillator, a bouncing ball, a particle influenced by gravity, a spring and mass, etc. These systems readily admit a phase-plane description and these phase portraits are very easy to draw.

Newton's law says that *force equals mass times acceleration* If we let x denote the postion of a particle, m denote its mass, and $-f$ denote the force, then Newton's law becomes:

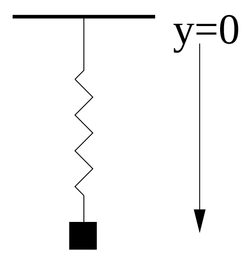
$$md^2x/dt^2 = -f \tag{1}$$

Some examples:

1. A falling object where only gravity works:

$$md^2x/dt^2 = -mg \tag{2}$$

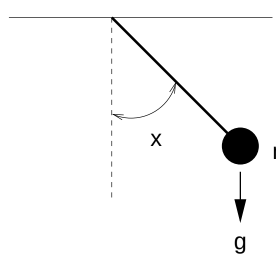
2. The frictionless spring



which has equations:

$$md^2y/dt^2 = -k(y-l) \tag{3}$$

3. The pendulum



which has equations:

$$md^2x/dt^2 = -mg\sin(x)/l \tag{4}$$

Recall from elementary physics that POTENTIAL ENERGY is the integral of force with distance and that KINETIC ENERGY is $mv^2/2$. The velocity, v is just dx/dt. Thus, if we differentiate the kinetic energy with respect to t, we get

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = m\frac{dx}{dt}\frac{d^2x}{dt^2} \tag{5}$$

Suppose we let $F(x) = \int_{x_0}^{x} f(y)dy$. Then

$$dF(x)/dt = f(x)dx/dt \tag{6}$$

Now multiply both sides of (1) by dx/dt and integrate this with respect to time. Then we get:

$$m\int (dx/dt)(d^2x/dt^2)dt = -\int f(x)(dx/dt)dt \tag{7}$$

From the discussion above, this implies that:

$$\frac{1}{2}m(dx/dt)^2 + F(x) = E \tag{8}$$

That is, the kinetic energy plus the potential energy is a constant, the TOTAL energy, E. The potential energy, $F(x)$ is just the integral of the force $f(x)$ with respect to x. The equation (8) can be used to sketch the phase planes of system like (1). Looking at (1) we can write it as a system of equations

$$dx/dt = v \tag{9}$$
$$dv/dt = -f(x)/m \tag{10}$$

As is the usual in this class, we want to draw the phase plane for this. As an example, lets look at (2). First, we note that the solution to (2) is

$$v = v_0 + gt \tag{11}$$
$$x = x_0 + v_0t - gt^2/2 \tag{12}$$

Rearranging this slightly, we see that x and v are related by the following

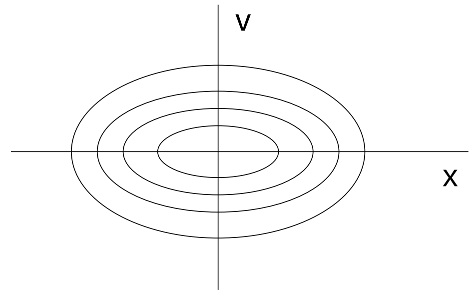
$$mv^2/2 + mgx = mv_0^2/2 + mgx_0 = E_0 \tag{13}$$

Using the formula (8) and the definition of $f(x)$ in (2) we could easily have deduced this relation without solving the equation. The curves $mv^2/2 + mgx = E_0$ are exactly what we want to plot in the phase-plane, thus, one sees that without solving the equations, the trajectories are precisely the curves given by (8). The trajectories for such an *integrable system* are called *integral curves.* For (2) these are a series of parabolas centered at the $x-$axis

Without actually solving (3) we see that the integral curves satisfy:

$$mv^2/2 + k(x-l)^2/2 = E \tag{14}$$

This equation defines a series of ellipses centered at $(l, 0)$



More generally, we want to sketch (8) which results in our sketching v versus x or:

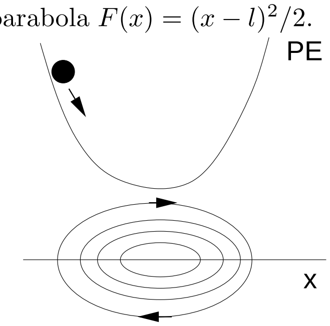
$$v = \pm 2\sqrt{\frac{E - F(x)}{m}} \tag{15}$$

As is clear, there are two parts to the curves corresponding to the positive and negative square roots, thus the phase-portrait is always mirror symmetric across the x axis.

The interesting behavior occurs at the local extrema of $F(x)$ for then $F'(x) = f(x) = 0$ so that local extrema of F are nothing more than equilibria for (9). The Jacobian matrix at a rest point, $(\bar{x}, 0)$ is:

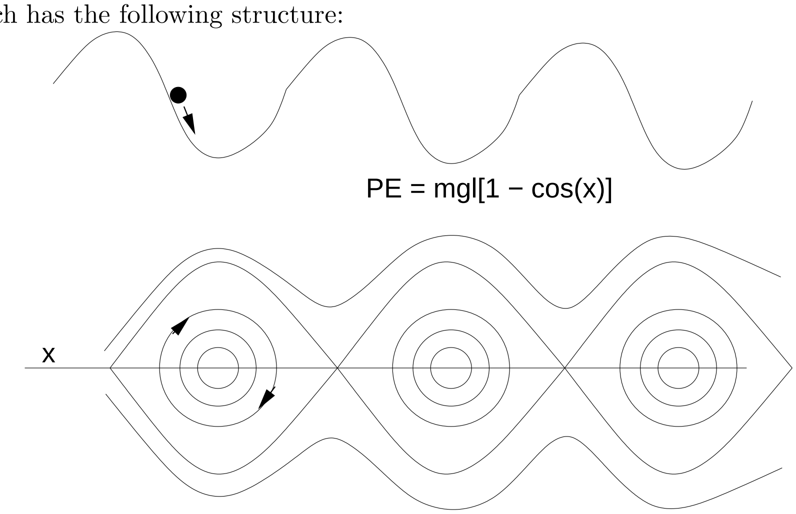
$$J = \begin{pmatrix} 0 & 1 \\ -f'(\bar{x})/m & 0 \end{pmatrix} \tag{16}$$

The trace of J is zero and the determinant is $f'(\bar{x})/m$. Thus if $f'(\bar{x}) > 0$ then the rest state is a *center* and if $f'(\bar{x}) < 0$ it is a *saddle.* Recall that $f'(x) = F''(x)$ so that if $f'(x) > 0$ then $\bar{x}$ is a local *minimum* of F and similarly if $f'(x) < 0$ $\bar{x}$ is a local *maximum* of F. Recall that $F(x)$ is the potential energy so that energy minima correspond to centers and energy maxima correspond to saddles. These two observations allow us to construct the phase-plane by simply plotting the potential energy. Consider again the spring equation. The potential energy is nothing more than the parabola $F(x) = (x - l)^2/2$.

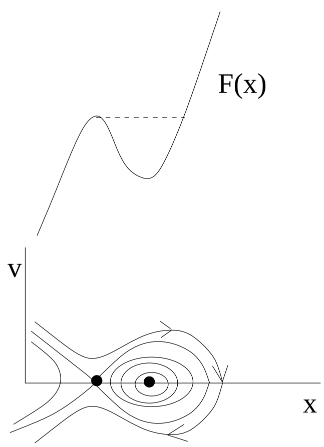


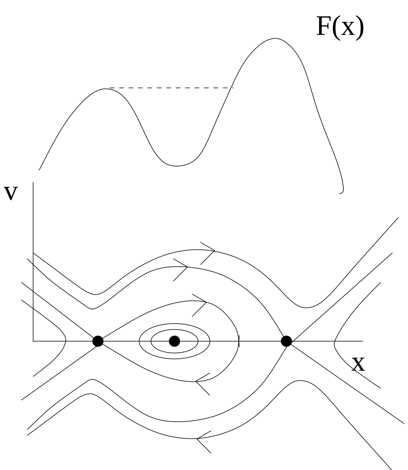
The idea is to view the potential energy as a hill and imagine a marble placed on that hill and allowed to roll. The initial velocity imparted along with its position on the potential landscape uniquely determine the energy and so allow us to sketch the integral curves for these models.

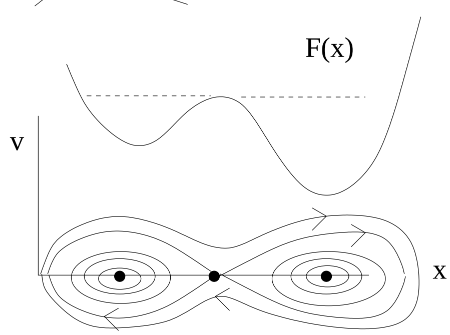
For the pendulum described by (4) we see that $F(x) = (mg/l)(1 - cos(x))$ which has the following structure:



Notice how there are trajectories joining the local maxima. Generally that will not happen because the local maxima do not have the same energy. Here are some additional examples.







2 Homework

1.
$$md^2x/dt^2 = -x(1-x)(x+a) \tag{17}$$
where $a > 1, a = 1, 0 < a < 1$

2. Given the following potential energies, sketch the phase-plane: