Exam 1 - Math 1270

Name ——————————————————

SS Number——————

Please answer each question.

1. Solve the following initial value problems
   (a) $y' - 3y/t = t^2$ with $y(1) = 2$
   (b) $t^2y'' - 4ty' + 6y = 0$ with $y(1) = 0$ and $y'(1) = 1$ (Note: this is not a series solution problem. However, the method for solving it is on page 3 of the series handout and also in exercise 6, p221 of the book.)
   (c) $y' = 1 + y^2$ with $y(0) = \alpha$. For each α compute the interval of existence $T_1 < t < T_2$ for the solution.
   (d) $y' + 2y/t = y^3$ with $y(1) = 1$. What is the interval of existence for the solution to this problem? (Note this is a Bernoulli equation - page 121 in the book.)
   (e) $y'' + 6y' + 8y = te^{-t}$ with $y(0) = 0, y'(0) = 1$.

2. Match the direction fields shown in the figure with the correct differential equation, $y' = f(t, y)$. Here are the choices for $f(t, y)$: (i) $t-1$, (ii) $1-y^2$, (iii) y^2-t^2, (iv) $1-t$, (v) $1-y$, (vi) t^2-y^2, (vii) $1+y$, and (viii) y^2-1. Explain your reasoning. Also, sketch a representative trajectory forward and backward in time with initial condition $y(0) = 3/2$. You do not need a computer to solve this.

3. Let $f(y) = (y-2)^2(y-3)(y-5)(y-6)$. Find all the equilibrium points to $y' = f(y)$ and determine their stability. In the (t, y) plane, sketch solutions with initial conditions $y(0) = 1.9, 2.1, 3, 5.5$.

4. Consider the differential equation:

$$(yx^2 - y)dy + (xy^2 - 2x^3 + x)dx = 0. \qquad (*)$$

   Show that there is an integral, $H(x, y)$ for the equation (*). Find $H(x, y)$. Using the method described by example 2.5.6 to plot integral curves for (*). A good window is $-2 < x < 2$ and $-2 < y < 2$.

5. If you attempt to tap your fingers in an alternating rhythm, as the frequency at which you tap increases, there is a sudden switch from alternate to synchronous tapping. Kelso and Haken proposed the following model for the phase-difference, ϕ, between your two fingers:

$$\frac{d\phi}{dt} = -\sin\phi - \lambda \sin 2\phi.$$

They suggest that as the frequency goes up, the parameter λ decreases. Draw the complete bifurcation diagram for this model. (That is, find all the equilibrium points and their stability.) Find any bifurcation points and determine the type of bifurcation, i.e., saddle-node, pitchfork, etc. Are there values of λ for which there are two stable equilibria? Does this model account for the experimental results at least qualitatively?

6. Here is a model for a simple laser involving the two variables, N the number of excited atoms and n the number of laser photons:

$$\begin{aligned} n' &= GnN - kn \\ N' &= -GnN - fN + p. \end{aligned}$$

The parameters are all positive and are G the gain coefficient, k decay rate of photons, f decay for spontaneous emission, and lastly, p the pump strength. Suppose that N is "fast" so that we can set $N' \approx 0$ that is solve

$$-GnN - fN + p = 0$$

for N in terms of n. Substitute this expression for N in terms of n into the n differential equation to get a single equation for n:

$$n' = GnN(n) - kn.$$

Find all the equilibrium points. Draw the bifurcation diagram. What kind of bifurcation is it? Don't forget to indicate stable and unstable branches.

7. The inverted pendulum. Jugglers and other performers are able to balance a long stick on their palm by jiggling it back an forth. The linear approximation to the angular displacement of the stick away from the vertical direction satisfies:

$$x'' + \mu x' + [-g/L + \beta \sin(ft)]x = 0. \qquad (**)$$

$\mu \geq 0$ is the friction, $L > 0$ is the length of the stick, and $g > 0$ is gravity. β is the amplitude of the jiggling and f is the frequency.

a. Set $\beta = 0$ and show that some solutions to (**) will exponentially grow.

b. Now suppose that β is not zero. Compute the Wronskian for a basic set of solutions, $x_1(t), x_2(t)$ satisfying

$$x_1(0) = 1, x_1'(0) = 0 \qquad x_2(0) = 0, x_2'(0) = 1.$$

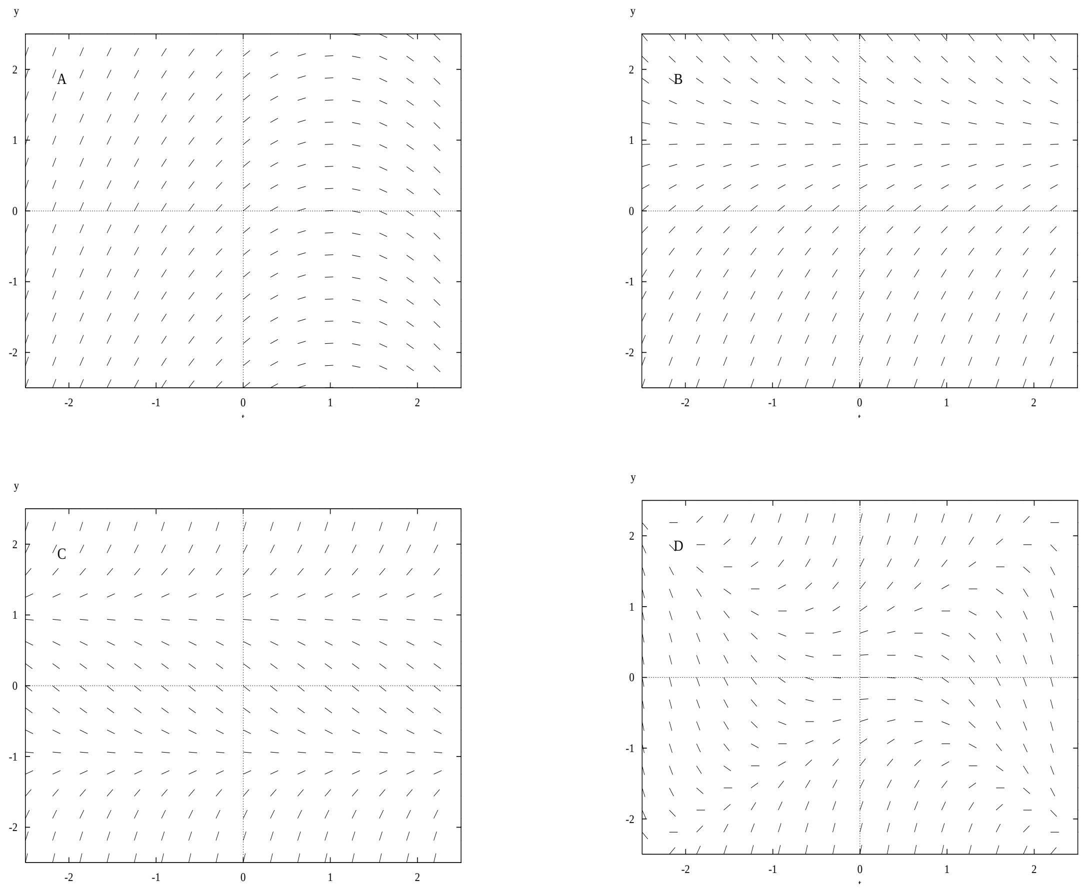
c. Suppose that $y_1(t), y_2(t)$ both exponentially decay to zero at t goes to infinity. Show that all solutions to $y(t)$ also decay to zero as t goes to infinity.

d. Set $L = 10$ meters $g = 9.8$, $f = 20$, and $\mu = 0.1$ Use the computer to solve this differential equation with initial conditions $x(0) = 1, x'(0) = 0$ for $\beta = 10, 20, 30, 40$. For what values of β do solutions appear to decay to zero? What is the smallest value of β for which solutions decay to zero? Hint: Write this as a system of 2 equations in order to use a solver:

$$x' = v, \quad , v' = -\mu v + (g/L + \beta \sin(ft))x$$

Here is an XPP file if you want:

```
x'=v
v'=-mu*v + ((g/L)+beta*sin(f*t))*x
par mu=0.1,beta=0,f=20,L=10,g=9.8
init x=1
@ total=100,xhi=100,yhi=5
done
```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B
C
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y
-2
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