

HOMEWORK 2

Due Sept 13

1. It is possible to prove existence directly using nothing more than calculus. Suppose that $f(t, x)$ is defined in $[t_0 - \alpha, t_0 + \alpha] \times B(\bar{a}, \beta)$ and bounded with bound, M . Let $b = \min(\alpha, \beta/M)$. f has a Lipschitz constant L . Let

$$y_{n+1} = a + \int_{t_0}^t f(s, y_n(s)) ds$$

with $y_0 = a$. Prove by induction that

$$|y_{n+1}(t) - y_n(t)| \leq \frac{ML^n(t - t_0)^{n+1}}{(n+1)!}$$

Conclude that

$$a + \sum_{n=0}^{\infty} [y_{n+1}(t) - y_n(t)] = y(t)$$

is uniformly convergent on $[t_0 - b, t_0 + b]$, that is

$$y(t) = \lim_{n \rightarrow \infty} y_n(t)$$

exists uniformly. Show that $y(t)$ is a solution to

$$y(t) = a + \int_{t_0}^t f(s, y(s)) ds.$$

2. Solve the integral equation:

$$x(t) = 1 + \int_0^t sx(s) ds.$$

Compute the first few terms for the Picard iteration and then use this to write the general n^{th} iterate. Show that the terms are same as the Taylor series expansion for your solution to the integral equation.

3. Find the maximal interval of existence to the ODE:

$$\dot{x} = x^2, \quad x(0) = a.$$