So we are very good for 

\[ \ln |v_i| < 0 \quad \text{for } i = 2, \ldots, N + \mu \mu \]

we have here that \( \lambda \ln |v_i| < 0 \)

even if \( \lambda > 0 \), Dani suggested Tau notation which is better!

In general:

\[ x_i(t+1) = \sum_{j=1}^{N} W_{ij} F(x_j(t)) \quad F : \mathbb{R}^m \to \mathbb{R}^m \]

when \( \sum_{j=1}^{N} W_{ij} = 1 \quad \forall i \)

and \( y(t+1) = F(y(t)) \) is a sequence (periodic or perhaps, not periodic)

Now to study stability of synchronous state. Let \( \nu_k \) be eigenvalue of \( M = (w_{ij}) \)

Then you must look at \( t \)

\[ \lambda_k = \ln \nu_k + \lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} \ln \left( |A(t)| \right) \]

where \( A(t) = D_x F(y(t)) \)

and \( |A(t)| \) is some matrix norm, for example:

\[ |A| = \sup_{v \neq 0} \sqrt{\sum_{i,j} |A_{ij}|^2} \]
This can be done best numerically since you cannot usually find $\gamma$.

If $U(t)$ is periodic, then

$\gamma \leq 0$ and so any coupling will synchronize.

CAVEAT: "OF THIS FORM"

Thus, a very special type of coupling.

Let me clarify this.

Suppose you have the two-dimensional map:

\[
\begin{align*}
U(t+1) &= f(U(t), V(t)) \\
V(t+1) &= g(U(t), V(t))
\end{align*}
\]

\[
\begin{pmatrix} U(t+1) \\ V(t+1) \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}(U(t), V(t))
\]

\[
\begin{pmatrix} U_i(t+1) \\ V_i(t+1) \end{pmatrix} = \sum_{j=1}^{N} \text{wij} \begin{pmatrix} f(U_j(t), V_j(t)) \\ g(U_j(t), V_j(t)) \end{pmatrix}
\]

Say that species $U$ and species $V$ migrate with exactly the same probability. In stead, the more general equation is
\[ u_{j}(t+1) = \frac{1}{N} \sum_{j=1}^{N} m_{ij} f(u_{i}(t), v_{j}(t)) \]

\[ v_{i}(t+1) = \sum_{j=1}^{N} m_{ij} g(u_{j}(t), v_{j}(t)) \]

So, if \( \delta_{ij} \neq m_{ij} \) then we cannot apply the previous result.

Indeed, we will see later with continuous differential equation, that different migration rates can have profound effects on synchrony.

---

Let's return to the map game and look at two different geometries of connectivity.

"All to All" \( m_{ij} = \begin{cases} 1 - \frac{c}{N-1} & i = j \\ \frac{c}{N-1} & i \neq j \end{cases} \)

Here \( c \) is the coupling rate.

\[
M = \begin{bmatrix}
1-c & \frac{c}{N-1} & \cdots & \frac{c}{N-1} \\
\frac{c}{N-1} & 1-c & \cdots & \frac{c}{N-1} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{c}{N-1} & \cdots & \frac{c}{N-1} & 1-c
\end{bmatrix}
\]

What are eigenvalues?
Aside:
\[ M = \begin{pmatrix} a & b & \cdots & b \\ b & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \cdots & b & a \end{pmatrix} = (a-b)I + b \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \ddots & \vdots \\ \vdots & \ddots & 1 \\ 1 & \cdots & 1 \end{pmatrix} \]

Eigenvalues of \[ \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \ddots & \vdots \\ \vdots & \ddots & 1 \\ 1 & \cdots & 1 \end{pmatrix} \] are \( N, 0 (N-1) \times \) (N times)

Eigenvalues of \( M \) are \( a-b + bN \) and \( a-b \) \( (N-1) \times \) (N times).

So for us: \( a = 1-c \), \( b = \frac{c}{N-1} \)

\( a-b+bN = 1-c - c \frac{c}{N-1} + \frac{cN}{N-1} = 1 \)

\( a-b = 1-c-c \frac{1}{N-1} \)

So \( \lambda_1 = 1 \) as usual

\( \lambda_2, \ldots, N = 1-c-c \frac{1}{N-1} \)

and Neurem says synchrony will be stable if

\[ \lambda + \ln \left| 1-c-c \frac{1}{N-1} \right| < 0 \]

For example, logistic map at \( \lambda \)

\[ u(t+1) = ru(t)(1-u(t)), \quad r > 3.9 \]

\( \lambda = 0.492 \) so need strong coupling
Example 2 for coupling:

Nearest neighbor coupling

\[ 0 < c \leq \frac{1}{2} \]

\[ M = \begin{bmatrix}
1 - 2c & c & 0 & \cdots & 0 & c \\
c & 1 - 2c & c & 0 & \cdots & 0 \\
0 & \cdots & c & 1 - 2c & c \\
\end{bmatrix} \]

Periodic ring of \( N \) elements.

Aside: (Always with \( N \) aside!)

Circulant matrices

Let \( \mathbf{c} = [c_0, c_1, c_2, \ldots, c_{N-1}] \)

Let \( M = \begin{bmatrix}
c_0 & c_1 & \cdots & c_{N-1} \\
c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
c_1 & c_2 & \cdots & c_{N-1} & c_0 \\
\end{bmatrix} \)

\( M \) is called a \textbf{circulant matrix}

\[ (M \mathbf{x})_i = \sum_{j=0}^{N-1} c_{i-j} x_j \]

or \textbf{convolution}
\((\hat{M} \vec{x})_i = \sum_{j=1}^{n} C_{j-i} \vec{x}_j\)

where we take \(j-i\) modulo \(N\)

e.g. if \(j-i = -3\) then add \(N\) to make it \(N-3\).

We have identified all elements to lie in a circle which is why it is called a circulant matrix.

Coupling from \(j\) to \(i\) depends only on \(j-i\).

---

Eigenvalues of circulant matrices are easy to find.

Let \(z^N = 1\), so \(z\) is an \(N\)th root of 1.

For example, \(N = 2\), \(z = \pm 1, 2\pi i, -2\pi i\)

\(N = 3\), \(z = e^{\frac{2\pi i}{3}}, \frac{1}{e^{\frac{2\pi i}{3}}}, e^{\frac{2\pi i}{3}}\)

\(N = 4\), \(z = +1, -1, i, -i\)

In general, \(z = e^{\frac{2\pi i k}{N}}\), \(k = 0, \ldots, N-1\)

\(\sin\theta \leq z^N = (e^{\frac{2\pi i k}{N}})^N = e^{\frac{2\pi i k N}{N}} = 1\)
Claim: \[ \mathbf{v} = \begin{bmatrix} 1 \\ z_n \\ z_{n-1} \\ \vdots \\ z_1 \end{bmatrix} \] is an eigenvector

Proof: 
\[
\mathbf{v} e \begin{bmatrix} A & D \\ D & A \end{bmatrix} t \begin{bmatrix} 1 \\ z_n \\ z_{n-1} \\ \vdots \\ z_1 \end{bmatrix} 
\]
\[
= \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} c_{kj} e \begin{bmatrix} A & D \\ D & A \end{bmatrix} t \begin{bmatrix} 1 \\ z_n \\ z_{n-1} \\ \vdots \\ z_1 \end{bmatrix} 
\]
\[
= \sum_{j=0}^{N-1} c_{j} e \begin{bmatrix} A & D \\ D & A \end{bmatrix} t \begin{bmatrix} 1 \\ z_n \\ z_{n-1} \\ \vdots \\ z_1 \end{bmatrix} 
\]
\[
\Rightarrow \mathbf{v} = \sum_{j=0}^{N-1} c_{j} e \begin{bmatrix} A & D \\ D & A \end{bmatrix} t \begin{bmatrix} 1 \\ z_n \\ z_{n-1} \\ \vdots \\ z_1 \end{bmatrix} 
\]

so \( \mathbf{v} \) is an eigenvalue

Proposition: If \( M \) is a circulant matrix with first row \( c_0, \ldots, c_{N-1} \), then the \( N \) eigenvalues are
\[ \mathbf{v}_k = \sum_{j=0}^{N-1} c_j e^{2\pi ik/j} \]
For nearest neighbor coupling
\[ c_0 = 1 - 2c \quad c_1 = c \quad c_{N-1} = c \quad c_N = 0 \]
so
\[ \nu_k = c_0 \cdot 1 + c_1 e^{\frac{-2\pi i k}{N}} + c_{N-1} e^{\frac{2\pi i k}{N}} \]
\[ = 1 - 2c + 2c \cos \frac{2\pi k}{N} \]
\[ = 1 - 2c \left( 1 - \cos \frac{2\pi k}{N} \right) \]
\[ k = 0, \ldots, N-1 \]
so the eigenvalues are all real.

Others are less than 1, but for \( N \) large, \( 1 - \cos \frac{2\pi k}{N} \)

is very close to zero.

The largest of these eigenvalues is
\[ \nu_{\text{max}} = 1 - 2c \cos \left( 1 - \cos \frac{2\pi}{N} \right) \quad (\nu_0 = 1 \text{ is largest}) \]
For \( N \) large, \( \cos x = 1 - \frac{x^2}{2} \quad (x \text{ small}) \)
so
\[ 1 - 2c \left( 1 - \cos \frac{2\pi}{N} \right) \approx 1 - 2c \frac{4\pi^2}{N^2} \]
and
\[ \ln \nu_{\text{max}} \approx -c \frac{4\pi^2}{N^2} \quad (N \text{ large}) \]
so we can only overcome a slightly positive 2
For example in our Logistic map example when $r = 3.9$ and $\lambda = .492$, we will not be able to synchronize large regions since $\ln \nu_{\max} = -c \frac{4\pi^2}{N^2}$ is small ($0 < c < \frac{1}{2}$).

**MONTE: Global (all-all) chaining is much better for synchrony than local.**

However, in many systems you can only access local information, such as Firefly. Thus $\lambda$ does not always be synchrony even for identical elements.

For this I will have you look at a bunch of other forms of matrices including Fe mammillary sparse cases.
Two metronomes or for now, pendulums

That rest on a board of mass \( M \)

Let \( X \) be center of mass of board

Let pendulums be at \( X + a_i \); \( i = 1, 2 \)

Let \( \theta_1, \theta_2 \) be angle of bobs + let

\( l = \text{length} \) + let \( m \) be mass of pendulums

Potential energy is just due to gravity (we will derive from later)

\[-mgL \left[ \cos \theta_1 + \cos \theta_2 \right] = PE\]

\[y_1 = -mgL \cos \theta_1, \quad y_2 = -mgL \cos \theta_2\]

\[x_1 = x + a_1 + L \sin \theta_1, \quad x_2 = x + a_2 + L \sin \theta_2\]

\[\dot{x}_1 = \dot{x} + \dot{\theta}_1 L \cos \theta_1, \quad \dot{x}_2 = \dot{x} + \dot{\theta}_2 L \cos \theta_2\]

\[\ddot{y}_1 = mgL \dot{\theta}_1 \sin \theta_1, \quad \ddot{y}_2 = mgL \dot{\theta}_2 \sin \theta_2\]
\[ \text{L.E.} = \frac{m}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{\theta}_1^2 + \dot{\theta}_2^2 \right) + \frac{M}{2} \dot{x}^2 \]

\[ = \frac{m}{2} \left( 2\dot{x}^2 + \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2x \dot{\theta}_1 \cos \theta_1 + 2x \dot{\theta}_2 \cos \theta_2 \right) + \frac{M}{2} \dot{x}^2 \]

Lagrangian = \text{Potential + K.E.} = L

Dynamics:

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \frac{\partial}{\partial \dot{x}} \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial \dot{x}} \right) \]

\[ L = mgL \cos \theta_1 + mgL \cos \theta_2 + \frac{2m+M}{2} \dot{x}^2 + \frac{mL^2}{2} \dot{\theta}_1^2 + mL \dot{\theta}_1 \dot{\theta}_2 \cos \theta_2 \]

\[ = \frac{2m+M}{2} \dot{x}^2 + mL \dot{\theta}_1 \dot{\theta}_2 \cos \theta_2 \]

\[ \frac{\partial L}{\partial \dot{x}} = \left( 2m+M \right) \dot{x} + mL \dot{\theta}_1 \dot{\theta}_2 \cos \theta_2 \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial L}{\partial x} = C \Rightarrow \]

\[ x = C - \frac{mL}{2m+M} \left( \cos \theta_1 + \cos \theta_2 \right) \]

\[ = \frac{C}{2m+M} \]

\[ 2m+M \]
\[
\begin{align*}
\frac{d^2 \theta_1}{d \theta_1} &= m g l^2 \dot{\theta}_1 + m l \dot{x} \cos \theta_1 \\
\frac{d^2 \theta_2}{d \theta_2} &= m g l^2 \dot{\theta}_2 + m l \dot{x} \cos \theta_2 - m g l \dot{\theta}_1 \sin \theta_1 \\
\end{align*}
\]

\[
\begin{align*}
ml^2 \ddot{\theta}_1 &= -m g l \sin \theta_1 - m l \cos \theta_1 \ddot{x} \\
ml^2 \ddot{\theta}_2 &= -m g l \sin \theta_2 - m l \cos \theta_2 \ddot{x} \\
\end{align*}
\]

\[
x = -\frac{m l}{2m + M} \left[ \dot{\theta}_1 \cos \theta_1 + \ddot{\theta}_2 \cos \theta_2 - \dot{\theta}_1 \sin \theta_1 - \ddot{\theta}_1 \sin \theta_1 \right] \\
ml \cos \theta_1 \ddot{x} = -\frac{m l^2}{2m + M} \left[ \dot{\theta}_1^2 \cos^2 \theta_1 + \dot{\theta}_2 \cos \theta_2 (\cos \theta_1^2 - \sin \theta_1 \sin \theta_2) - \dot{\theta}_1 \sin \theta_1 \cos \theta_1 \sin \theta_2 \right] \\
ml \cos \theta_2 \ddot{x} = -\frac{m l^2}{2m + M} \left[ \dot{\theta}_2 \cos \theta_2 + \dot{\theta}_1 \cos \theta_2 \cos \theta_1 - \dot{\theta}_1 \sin \theta_1 \cos \theta_2 \sin \theta_1 \cos \theta_2 \right] \\
\frac{m l^2}{2m + M} \left[ \begin{array}{c}
\dot{\theta}_1^2 \cos^2 \theta_1 \\
\dot{\theta}_2 \cos \theta_2 (\cos \theta_1^2 - \sin \theta_1 \sin \theta_2) \\
\dot{\theta}_1 \sin \theta_1 \cos \theta_1 \sin \theta_2 \end{array} \right] \\
\frac{m l^2}{2m + M} \left[ \begin{array}{c}
\dot{\theta}_2 \cos \theta_2 + \dot{\theta}_1 \cos \theta_2 \cos \theta_1 \\
\dot{\theta}_1 \sin \theta_1 \cos \theta_2 \sin \theta_1 \cos \theta_2 \end{array} \right]
\]

Thus \( \text{invertible} \) so we can solve for \( \dot{\theta}_1 \) \( \dot{\theta}_2 \).
\[ L = \frac{1}{2} (M + 2m) \dot{x}^2 + m l \lambda (\dot{\theta}_1 \cos \theta_1 + \dot{\theta}_2 \cos \theta_2) \]
\[ + \frac{1}{2} ml^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) + mgl (\cos \theta_1 + \cos \theta_2) \]
\[ + k \sum \frac{\Delta x}{\Delta x} \Rightarrow \]
\[ \frac{d}{dt} \frac{dx}{dt} = \frac{dx}{\Delta x} \Rightarrow \]
\[ \frac{d}{dt} \left[ (M + 2m) \dot{x} + ml (\dot{\theta}_1 \cos \theta_1 + \dot{\theta}_2 \cos \theta_2) \right] = -kx \]

$B$ is damping

We can rewrite this as

\[ (M + 2m) \ddot{x} + B \dot{x} + kx = -ml \left[ \sin \theta_1 + \sin \theta_2 \right] \]

\[ \frac{d}{dt} \frac{dx}{dt} = \frac{dx}{\Delta \theta_k} \Rightarrow \]
\[ ml^2 \ddot{\theta}_k + mgl \sin \theta_k = -ml \dot{x} \cos \theta_k - b ml \dot{\theta}_k \]
\[ + ml^2 \dot{\theta}_k \]

Here $b$ is friction or damping.

$F_k$ is restoring force for the clock.
Let $Y = \frac{v}{l}$, $Z = \frac{\sqrt{g}}{l}$. Then we get

$$\theta_k'' + 2\pi \theta_k' + \sin \theta_k = -Y'' \cos \theta_k + f_k$$

$$Y'' + 2\pi Y' + \frac{\pi^2}{l^2} Y = -M (\sin \theta_1 + \sin \theta_2)^{\prime \prime}$$

Where $Y = b\sqrt{g/l}$, $\Gamma = B\sqrt{g/l}$, $\Omega^2 = \frac{K}{M+2m}$

$M = \frac{m}{(M+2m)}$

This is a messy non-linear equation.

However, if $\theta_0, \pi \approx 0$, we can approximate it by a linear equation.

Before doing so, we need to determine $\Gamma$.

Perfunctory force of $\pi$ (clockwise; otherwise, it will just damp to zero!)

We introduce a simple mechanism. Whenever the pendulum reaches a threshold angle $\pm \varphi$, its angular velocity reverses direction and its magnitude changes according to

$$|\theta_k'| \rightarrow (1-c) |\theta_k'| + \varepsilon$$

$c, \varepsilon$ are small. Since $M > m$, the impulse has negligible effect on the platform.

The linear approximation is

$$\theta_1'' + 2\pi \theta_1' + \theta_1 = -Y'' + f_1$$

$$\theta_2'' + 2\pi \theta_2' + \theta_2 = -Y'' + f_2$$

$$Y'' + 2\pi Y' + \frac{\pi^2}{l^2} Y = -M (\theta_1'' + \theta_2'')$$
We rewrite these as

\[ \theta''_1 + \gamma'' = \zeta_1 = -28\theta'_1 + \theta_1 f_1 \text{ Note } m < 1 \]

\[ \theta''_2 + \gamma'' = \zeta_2 = -28\theta'_2 - \theta_2 f_2 \]

\[ \gamma'' + m \theta''_1 + m \theta''_2 = \zeta_3 = -2 \Gamma \gamma' - \delta^2 \gamma \]

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
m & m & 1
\end{bmatrix}
\begin{pmatrix}
\theta''_1 \\
* \theta''_2 \\
\gamma''
\end{pmatrix} =
\begin{pmatrix}
\zeta_1 \\
\zeta_2 \\
\zeta_3
\end{pmatrix}
\]

\[
\begin{align*}
\theta''_1 &= \frac{\zeta_1 + m(\zeta_2 - \zeta_1) - \zeta_3}{1 - 2m} \\
\theta''_2 &= \frac{\zeta_2 + m(\zeta_1 - \zeta_2) - \zeta_3}{1 - 2m} \\
\gamma'' &= \frac{\zeta_3 - m(\zeta_1 + \zeta_2)}{1 - 2m}
\end{align*}
\]

Thus linear between impulses. I tried solving it but only found out of phase locking.
Before moving on, I want to look at a variation with nonlinear forcing terms:

\[ \dot{\theta}'' + \gamma (\dot{\theta}^2 - \dot{\theta}_0^2) \dot{\theta}' + \theta = 0 \]

This cannot self-sustain. Suppose we consider:

\[ \dddot{\theta} + \gamma (\dot{\theta}^2 - \dot{\theta}_0^2) \dot{\theta}' + \theta = 0 \]

For small crease \( \theta = \theta_0 \phi \)

\[ \dot{\theta}_0 \dot{\phi}'' + \gamma \theta_0^2 (\dot{\theta}^2 - 1) \dot{\phi}' + \theta_0 \dot{\phi} = 0 \]

\[ \Rightarrow \dddot{\phi} + \gamma (\dot{\phi}^2 - 1) \dot{\phi}' + \theta = 0 \]

This is self-sustained - not oscillator.

Exact same as before but:

\[ z_1 = -\gamma (\dot{\theta}^2 - \dot{\theta}_0^2) \dot{\theta}' \text{ - } \theta \]

\[ z_2 = -\gamma (\dot{\theta}_0^2 - \dot{\theta}^2) \dot{\theta}_0' \text{ - } \theta_0 \]

Later we will explore this in detail using some weakly nonlinear perturbation analysis.
Flows

So, what can we say about Flows

Review of Limit Cycles & Stability

- Linear systems

\[ \dot{x} = Ax \quad A \text{ is constant} \quad (1) \]

\[ x(t) = e^{tA}x(0) \]

\[ e^{tA} = \text{exponential of matrix}. \quad \text{if } AV = AV \]

Then \( x = Ve^{t} \) is a solution.

\[ \lambda \in \text{spectrum of } A \]

If some \( \lambda \in \sigma(A) \) have negative real part then all solutions \( x(t) \to 0 \) decay to 0 at \( t \to -\infty \). \( x = 0 \text{ N. Assym stable} \)

- Linear periodic systems

\[ \dot{x} = A(t)x \quad A(t+T) = A(t) \quad (2) \]

Floquet Theorem Here

let \( \Phi(t) \) be the fundamental matrix for (2)

That is \( \dot{\Phi}(t) = A(t)\Phi(t) \) and suppose

\[ \Phi(0) = I \]

Then \( \Phi(t) = e^{\int_{0}^{t} \phi(t) dt} \)

Then \( \Phi(t) = e^{\int_{0}^{t} \phi(t) dt} \)

\[ \Phi(0) = I \Rightarrow \Phi(t) = e^{\int_{0}^{t} \phi(t) dt} \]

\[ \Phi(T) = e^{\int_{0}^{T} \phi(t) dt} \]

\[ \Phi(T) = e^{\int_{0}^{T} \phi(t) dt} \]
\[ \Delta(nT) = \rho(nT) e^{nTB} \]

Call \( M = e \)

\[ M^n \to 0 \quad \text{as} \quad n \to \infty \quad \text{if all eigenvalues of} \ M \ \text{are in unit circle.} \]

We call \( \rho \in \Sigma(M) \) a **Floquet Multiplier**

If we write \( \rho = e^{\lambda T} \) then \( \lambda \) is called a **Floquet Exponent**

These are defined up to multiplies of \( 2\pi i \).

If \( \text{Re} \lambda < 0 \iff |\rho| < 1 \iff \rho^n \to 0 \)

**Theorem:** If all Floquet exponents have negative real parts then all solutions to (2) decay to 0 as \( t \to \infty \).

**Theorem:** If there is a nontrivial periodic solution to (2) then there must be at least one multiplier \( \rho \) s.t. \( |\rho| = 1 \).

Autonomous systems & L.C.

(consider \( \dot{U} = FU \))

\[ \text{If } \dot{U}(t) + U_0(t + T) = U(t+T) \text{ is } T\text{-periodic solution.} \]
Write \( u(t) = u_0(t) + y \) where \( y \) is small.

Then
\[
\dot{u} = \dot{u}_0 + \dot{y} = F(u_0(t)) + \dot{y}(t) \\
\approx D_u F(u_0(t)) y(t) + u_0(t) + O(1/y^2)
\]

\( D_u F(u_0(t)) y(t) \equiv A(t) \) is \( T \)-periodic.

So... What happen to \( y(t) \).

Remark \( \dot{y} = A(t) y \) has a multiplier \( \rho = 1 \).

Proof \( \frac{d}{dt} u_0 = F(u_0(t)) \)

Differentiating:
\[
\frac{d}{dt} u_0(t) = \frac{d^2 u_0}{dt^2} - D_u F(u_0(t)) \frac{du_0}{dt} = A(t) \dot{u}_0
\]

Thus \( T \)-periodic solution \( u_0 \) s.t. \( \dot{y} = A(t) y \)

\[ \Rightarrow \rho = 1 \]

We say a limit cycle is asymptotically stable if remaining Floquet multipliers are inside unit circle.

Example
\[
\dot{x} = x(1-x^2-y^2) - y \\
\dot{y} = y(1-x^2-y^2) + x
\]

\( x = \cos t \) \( y = \sin t \) is limit cycle, exponent

Exercise: Find the nonzero Floquet exponent
Helpful hint: Let \( \Phi(t) = \mathbf{A}(t) \) be a fundamental matrix then
\[
\int_0^T \text{tr} \mathbf{A}(s) ds \quad \rho_1, \rho_2, \ldots, \rho_n = e^{-\frac{\int_0^T \text{det} \Phi(t) dt}{n}}
\]
( This is a well-known identity from ODE’s -- I think it’s called Abel’s formula? )

With this background, we are now ready to study coupled systems:

\[
\begin{align*}
X_1 &= F(X_1) + \mathbf{K}(X_2-X_1) \\
X_2 &= F(X_2) + \mathbf{K}(X_1-X_2)
\end{align*}
\]

\(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) \(\mathbf{K} \in \mathbb{R}^{n \times n}\)

\(\dot{u}_0 = F(u_0(t))\) has a \(T\)-periodic \(L^2\) solution

\(X(t) = X_2 = u_0(t)\) is a synchronous solution

Write \(X_j = u_0(t) + y_j(t)\)

\(\Rightarrow\)

\[
\begin{align*}
\dot{y}_1 &= A(t) y_1 + \mathbf{K}(y_2-y_1) \\
\dot{y}_2 &= A(t) y_2 + \mathbf{K}(y_1-y_2)
\end{align*}
\]

where \( A(t) = DuF(u_0(t))\)
\[
\begin{pmatrix}
    y_1 \\ y_2
\end{pmatrix} =
\begin{bmatrix}
    A(t) & 0 \\
    0 & A(t)
\end{bmatrix}
\begin{pmatrix}
    y_1 \\ y_2
\end{pmatrix} +
\begin{bmatrix}
    -k & +k \\
    k & -k
\end{bmatrix}
\begin{pmatrix}
    y_1 \\ y_2
\end{pmatrix}
\]

Notationally, it is sometimes nice to write this as the Kronecker product of matrices:

\[
\begin{bmatrix}
    A(t) \otimes I + k \otimes Q
\end{bmatrix}
\]

\[
I = \begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix}
\]

\[
A(t) \otimes I = \begin{bmatrix}
    A(t) \cdot 1 & 0 \\
    A(t) \cdot 0 & A(t) \cdot 1
\end{bmatrix}
\]

\[
Q = \begin{bmatrix}
    -1 & 1 \\
    1 & -1
\end{bmatrix}
\]

"adjacency matrix"

Oops

\[
I \otimes A(t) + Q \otimes K
\]

Where \( Q = \begin{bmatrix}
    -1 & 1 \\
    1 & -1
\end{bmatrix} \) = adjacency matrix

\[
I = \begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix}
\]

\[
I \otimes A(t) = \begin{bmatrix}
    1 \cdot A & 0 \cdot A \\
    0 \cdot A & A \cdot A
\end{bmatrix} = \begin{bmatrix}
    A & 0 \\
    0 & A
\end{bmatrix}
\]

(1) \(\rightarrow\) (2) adjacency matrix

Who is coupled to who
\[
\begin{align*}
\text{Let } & \quad z = y_1 - y_2 \quad \text{ } w = z_1 + z_2 \\
\dot{z} & = \dot{y}_1 - \dot{y}_2 = A(t) y_1 - A(t) y_2 + k(y_2 - y_1) \\
& \quad - k(y_1 - y_2) \\
& = A(t) z - 2k z \\
\dot{w} & = A(t) w
\end{align*}
\]

As we did earlier, we have reduced the system to two equations of smaller dimension.

\[
\begin{align*}
U \quad z &= (A(t) - 2k) z(t) \\
\end{align*}
\]

\( w = A(t) w \)

\( z \) is just the single isolated limit cycle. Since \( \mu \) is stable, we know that \( (2) \) has all multipliers inside the unit circle, except for \( 1 \) which is due to \( \mu \).

So we can say that synchrony will be stable if all solutions to \( (1) \) decay to zero as \( t \to \infty \) since

\[
\begin{align*}
\dot{z}(t) & \to 0 \implies y_1 - y_2 \to 0 \implies y_1 - y_2 \to 0 \\
\text{as } t \to \infty & \implies \text{synchrony!}
\end{align*}
\]

So it is hard to say what happens with equation \( (1) \)

Rem: Notice that the eigenvalues of \( \dot{Q} \) are \( 0 \) and \( -2 \) ! (Sound familiar??)
Suppose \( k = 0 \), that is \( k \) is a scalar multiple of the identity. Then we can draw some conclusions.

Let \( \mathbf{z}(t) \) satisfy

\[
\dot{\mathbf{z}}(t) = A(t) \mathbf{z}(t)
\]

and suppose \( \mathbf{z}(t) \).

Then we can see that

\[
\eta(t) = e^{-2\sigma t} \mathbf{z}(t)
\]
satisfies:

\[
\frac{d\eta}{dt} = -2\sigma e^{-2\sigma t} \mathbf{z}(t) + e^{-2\sigma t} \dot{\mathbf{z}}(t) = -2\sigma \eta + e^{2\sigma t} A(t) \mathbf{z}(t)
\]

\[
= A(t) \eta(t) - 2\sigma \eta(t)
\]

That is \( \eta(t) \) solves (1) (only when \( k = 0 \)).

So if \( \mathbf{z}(t) \) is periodic then \( \eta(t) \) will decay for \( \sigma > 0 \) and grow exponentially if \( \sigma < 0 \).

If \( \mathbf{z}(t) \) decays then so will \( \eta(t) \) as long as \( \sigma > 0 \).

From (1) we can conclude that scalar "dissipative" coupling of oscillators will always synchronize them.
The main interest comes from non-scalar coupling case.

(N.B. for chaotic systems, the matrix \( A(t) \) is not periodic and so you must look at the long term growth of

\[
\dot{y} = A(t) y \tag{3}
\]

When you choose, there is always a solution to (3) such that \( |y(t)| = Ce^\lambda t \) where \( \lambda \geq 0 \). This number \( \lambda \) is called the maximal Lyapunov exponent.

The variational equation for the coupled system with scalar diffusive coupling is

\[
\dot{z} = A(t) z -2\omega z \tag{4}
\]

so if \( 2\omega > \lambda \) then all solutions to (4) will decay and synchrony will be stable.

With scalar coupling of oscillators, magnitude does not matter but with chaos it does (!!!)

Non-scalar coupling is much more interesting

So how do we analyse it?

End
Recovar-Carroll Master Stability equation.

\[ \dot{Z} = A(t) Z + (\alpha + i \beta) K \bar{Z} \]

Find the regions in \((\alpha, \beta)\) where \(Z(t) \to 0\) as \(t \to \infty\).

If we write \(Z = R + i S\) then

\[ \frac{dR}{dt} = A(t) R + \alpha K R - \beta K S \]
\[ \frac{dS}{dt} = A(t) S + \alpha K S + \beta K R \]

For periodic \(A(t)\), you can just integrate this, or you can compute the monodromy matrix as a function of \(0 + (\alpha, \beta)\).