If \( r, s \) are non-zero, then
\[
\alpha_0 + \alpha_1 r + \alpha_2 s^2 = 0 \quad \alpha_0 + \alpha_1 s + \alpha_2 r^2 = 0
\]
As long as \( \alpha_1 \neq \alpha_2 \Rightarrow r^2 = s^2 \),

I will leave as an exercise the stability theory for these patterns.

Bifurcations on a lattice

Consider a planar equation, such as RD or Neural networks, eg

\[
U_t = F(U, \nabla U) + D \nabla^2 U
\]

\( U : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^n \). As usual let \( U = 0 \) be a steady state homogeneous solution linearize to get

\[
M_n = (A - D k^2)^{\frac{1}{2}} \quad \text{where} \quad U = e^t \begin{pmatrix} e^{i(k_1 x + k_2 y)} \\ \cdots \end{pmatrix}
\]

If \( M_n \) has a zero eigenvalue \( k^2 = k_1^2 + k_2^2 \). If \( M_n^* \) has, say a zero eigenvalue, then we lose stability to \( k^2 \). But there are infinitely many choices for \( k_1, k_2 \). Null space has a continuum of members. Thus to apply our nice theory, we need to make the nullspace finite dimensional. One standard way to do this is to restrict the patterns on a lattice, much too doubly periodic pattern in a plane. There are 3 ways to tessellate the plane with regular polygons, the square, rhombus, and hexagon. These are all defined by the angles between two vectors such that

\[
\|k_1\| = \|k_2\| = k
\]

(a) \( k_1, k_2 = 0 \) squares

(b) \( k_1, k_2 = k^2 \left( \cos \frac{\pi}{3}, \cos \frac{2\pi}{3} \right) \) hexagons

(c) \( k_1, k_2 = 0, k^2 \left( \cos \frac{\pi}{3}, \cos \frac{\pi}{3} \right) \) rhombi
Each has different symmetries, and hence different amplitude equations which we will derive. Null space is 4 dimensional for rhombus/square and 6-dimensional for hexagon.

We want to use symmetry to derive equations.

Symmetry of our equations are reflection in space. Let $e^{i\theta}$ be the rotation.

\[
\text{SQUARE: } z_1 e^{i\theta}, z_2 e^{i\theta}, z_3 e^{i\theta}, z_4 e^{i\theta}
\]

We want real solutions so have $z_3 = \overline{z_1}$, $z_4 = \overline{z_2}$, let call term $(\overline{z_2}, \overline{z_3})$

In reflection $z_1 \rightarrow z_3$, reflection $z_2 \rightarrow z_4$

We have
\[
\begin{align*}
\tilde{z}_1 &= F_1(z_1, z_2, z_3, z_4) \\
\tilde{z}_2 &= F_2(z_1, z_2, z_3, z_4) \\
\tilde{z}_3 &= F_3(z_1, z_2, z_3, z_4) \\
\tilde{z}_4 &= F_4(z_1, z_2, z_3, z_4)
\end{align*}
\]

(a) $F_3 = F_1$, $F_4 = F_2$, reality condition
(b) $F_1(z_1, z_2, z_3, z_4) = F_3(z_3, z_2, z_1, z_4)$ reflection
$F_2(z_1, z_2, z_3, z_4) = F_4(z_1, z_2, z_3, z_4)$ reflection
$F_3(z_1, z_2, z_3, z_4) = F_2(z_3, z_1, z_4, z_2)$ rotation

Thus we can express all equations in terms of $F_1$.
\[ F_2(z_1, z_2, z_3, z_4) = F_1(z_2, z_1, z_4, z_3) \]
\[ F_3(z_1, z_2, z_3, z_4) = F_1(z_3, z_2, z_1, z_4) \]
\[ F_4(z_1, z_2, z_3, z_4) = F_1(z_2, z_1, z_4, z_3) \]

We next use the translation invariance:
\[ T \{ e^{i\{g_1, g_2, \ldots, g_n\}} \} = e^{i\{g_1, g_2, \ldots, g_n\}} \]
\[ g = (g_1, g_2) \]

Let \( \omega \) set \( z_1 = z_1, z_2 = z_2, z_3 = z_3, z_4 = \bar{z}_4 = \bar{z}_3 = \bar{z}_3 = \bar{z}_3 \)

\[ F_1 = \sum \alpha_{pqrs} z^p \bar{z}^q \bar{w}^r \bar{w}^s \]

\[ T \{ F_1 \} = e^{i\{g_1, g_2, \ldots, g_n\}} F_1 \]

\[ e^{i\{g_1, g_2, \ldots, g_n\}} F_1 = F_1(e^{i\{g_1, g_2, \ldots, g_n\}} z, e^{i\{g_1, g_2, \ldots, g_n\}} \bar{z}) \]

\[ \Rightarrow e^{i\{g_1, g_2, \ldots, g_n\}} = e^{i\{g_1, p, q, r, \ldots, s\}} \]

\[ \Rightarrow p = r + 1 \quad q = s \]

So
\[ \sum \alpha_{pqrs} z^p \bar{z}^q \bar{w}^r \bar{w}^s \]

To lowest order
\[ \alpha_{00} z + \alpha_{10} z^2 \bar{z} + \alpha_{01} \bar{z} \bar{w} \]

\[ F_1(z, \bar{z}, w, \bar{w}) = \alpha_{00} z + \alpha_{10} z^2 \bar{z} + \alpha_{01} \bar{z} \bar{w} \]

\[ \text{reflection} \quad F_3 = F_1 \] but also reflection so \( \alpha_{00}, \alpha_{10}, \alpha_{01} \) are all real since \( F_1(z, z, w, w) = F_1(z, \bar{z}, w, \bar{w}) \)

Thus
\[ z \bar{z} = z(z_{00} + \alpha_{10} z^2 \bar{z} + \alpha_{01} \bar{w}) \]
\[ \bar{w} \bar{w} = w(\alpha_{00} + \alpha_{10} w^2 + \alpha_{01} z^2) \]

What are \( z, \bar{z}, w, \bar{w} \) possible solutions? \( (\alpha_{00}, \alpha_{10}, \alpha_{01}) \)

Set \( z = re^{i\theta}, \ w = se^{ip} \)
\[ z = r(e^{i\theta} \alpha_{00} + \alpha_{10} r^2 + \alpha_{01} s^2) \]
\[ s = s(e^{i\theta} \alpha_{00} + \alpha_{10} s^2 + \alpha_{01} r^2) \]
A trivial null: $r = s = 0$, $(r = 0, s = \sqrt{\frac{-\lambda_{00}}{\lambda_{10}}})$

Vertical nulls:

$(r = \sqrt{\frac{-\lambda_{00}}{\lambda_{10}}}, s = 0)$

$(r = s = \sqrt{\frac{-\lambda_{00}}{\lambda_{10} + \lambda_{01}}})$

Check:

WLOG, let assume $\lambda_{00} > 0$. (Each null $\lambda_{00}(\lambda)$ starts so as $\lambda \rightarrow 0$ when $\lambda_{00} \neq 0$ so $\lambda_{00} = \lambda_{00}(0)$. Assume $\lambda_{00} > 0$ so as $\lambda \rightarrow 0$ curve $(0, 0)$ becomes unstable.

Let's linearize:

Let make notation simpler: $\lambda_{00} = \lambda$, $\lambda_{10} = -b$, $\lambda_{01} = -c$

so equation are:

$r = \lambda - b r^2 - c s^2$

$s = \lambda s - 6 b^2 - c r^2$

$D_f = \begin{pmatrix} \lambda - 6 b r^2 - c s^2 & -2 c r s \\ -2 c r s & \lambda - 6 b^2 - c r^2 \end{pmatrix}$

$(b+c) r^2 = \lambda$ when $r, s$ are non-zero

$\Rightarrow r^2 = \frac{\lambda}{b+c}$

These are called "spurious".

$r=0, s \neq 0$ or $s=0, r \neq 0$ are called "rolls".

If rolls $\leftrightarrow$ spurious have same structure and existence.

$\text{Square } D_f = \begin{pmatrix} -\frac{2 b \lambda}{b+c} & -\frac{2 c \lambda}{b+c} \\ -\frac{2 c \lambda}{b+c} & -\frac{2 b \lambda}{b+c} \end{pmatrix}$

$r^2 = s^2 = \frac{\lambda}{b+c}$

Eigenvalues are $-2 \frac{b+c}{b+c}$

$r^2 = \frac{\lambda}{b} s = 0$

Rolls $D_f = \begin{pmatrix} -2 \frac{\lambda}{b} & 0 \\ 0 & \lambda(1 - \frac{c}{b}) \end{pmatrix}$
From this we deduce the following:

For example, $b < 20$

Note rhombic lattice is similar. How do we compute coefficients in these problems?

I will do more R^2 equation but in general,

I will do Mi fairly abstractly.

Let's consider the following problem:

$$U_t = L_0 \nu + \lambda L_1 U + Q(U,U) + C(U,U,U) + \ldots$$

Where $L_0$, $L_1$ is a linear operator on some Banach space with planar symmetries, $Q$ any quadratic, $C$ are cubics, etc.

I will do the rhombic and square lattices simultaneously.

We rescale space so eigen vectors are

$$e_3 = (\cos \theta + \sin \theta y) e_3$$

$$e_3 = \theta = \frac{\pi}{4} \text{ is square}$$

$$\theta = \frac{2\pi}{3} \text{ or } \frac{\pi}{3}$$
\[ \text{Lo}(\nu e^{i_0 \cdot \vec{x}}) = \left( \text{Lo}(1 \nu) \right) V \left( e^{i_0 \cdot \vec{x}} \right) \]

because of Euclidean invariance. Similarly for \( V \) and \( Q_1 + C \) (since there are bi (tri) linear).

As we usually do write

\[ u(x, t) = e^{i_0 \cdot \vec{x}} \left[ \text{Lo}(1 \nu) \right] V e^{\frac{i_0 \cdot \vec{x}}{2}} + \nu_2 e^{\frac{i_0 \cdot \vec{x}}{2}} + \nu_3 e^{\frac{i_0 \cdot \vec{x}}{3}} + \ldots \]

with \( \nu_2, \nu_3 \) functions of \( T = \frac{\vec{x}^2}{2} \) and \( \nu = \frac{\vec{x}^2}{2} \).

3 order

\[ \nu_2 = 0 \quad \text{Lo}(1)^2 = 0 \]

Write \( c = \cos \theta \)

Let \( \text{Lo} (1) \eta = 0 \quad \eta^2 = 1 \)

\( \eta \)

3 order

\[ 0 = \text{Lo} U^2 + Q(\nu_2, \nu_2) \left[ \nu^2 e^{i(c \pm 2 \nu_2)} + \nu^2 e^{i(c \pm 2 \nu_2)} \right] \]

\[ + 2 \nu e^{i(c \pm 2 \nu_2)} \left[ e^{i(c \pm 2 \nu_2)} + e^{i(c \pm 2 \nu_2)} \right] \]

\[ + e^{i(c \pm 2 \nu_2) \nu_2} + e^{i(c \pm 2 \nu_2) \nu_2} \]

\[ \text{Lo} e^{i \cdot \vec{V}} = \left( \text{Lo}(\nu_2) \right) e^{i \cdot \vec{V}} \]

\( \text{Lo}(4) \) covers \( \nu^2, \nu^2 \)

\( \text{Lo}(0) \) covers \( 2 \nu, \nu \)

\( (c-\nu_2)^2 + \nu_2^2 = c^2 + \nu_2^2 + 1 - 2 c \nu_2 = 2(1-c) \text{Lo}(2c \nu_2) \)

\( (c+\nu_2)^2 + \nu_2^2 = 2(c+\nu_2) \)

All other new vertices by hypotenuse so we can solve quadratic part.

Note what if \( c = \frac{1}{2} \) ? UH OH but that is hexagonal lattice! So \( c = \frac{1}{2} \)
Thus we get

\[ u_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} i^n(x + \lambda x) \int_0^\infty \frac{d\lambda}{\lambda} \int_0^{\infty} \frac{d\tau}{\tau} \left( e^{i\tau} e^{i\lambda x} \right) \]

where

\[ \left( L(4) V_{20} + Q(3,3) \right) = 0 \quad \text{(independent of } \theta) \]
\[ \left( L(0) V_{00} + Q(3,3) \right) = 0 \quad \text{(independent of } \theta) \]
\[ \left( L(2) \cos \theta \right) V_{11} + Q(3,3) = 0 \quad \text{dependent of } \theta \]
\[ \left( L(2) \sin \theta \right) V_{11} + Q(3,3) = 0 \quad \text{dependent of } \theta \]

So omit cubic terms

\[ (z_+ e^{i\lambda} + \bar{w} e^{i\lambda}) \bar{z} = L_0 u_3 + \lambda L_1(1 \bar{z} e^{i\lambda} + \bar{w} e^{i\lambda}) \]

\[ + \text{ Note } L_1(1 \bar{z} e^{i\lambda} + \bar{w} e^{i\lambda}) + L_1(3,3,3) \left[ 3 \bar{z} \bar{w} e^{i\lambda} + \bar{w} \bar{w} \bar{z} e^{i\lambda} \right] \]

Apply Frobenius alternative + get

\[ z_+ = z \left[ \sum_{n=0}^{\infty} a_n + \bar{w} \bar{w} a_0 \right] \]

\[ a_0 = \eta \cdot L_1(1 \bar{z}) = 0 \]

\[ a_1 = \eta \left[ 3 C(3,3,3) + 4 Q(3,3,3) + 2 Q(3,3,3) \right] \]
\[ a_0 = \eta \left[ 6 C(3,3,3) + 4 Q(3,3,3) + 4 Q(3,3,3) \right] \]

This is pretty cool - only \( \bar{w} \bar{w} \) terms in \( z \) equation contain lattice dependence.

Also if no quadratic terms, then no lattice dependence and also \( \eta \) here will never be

Thus \( \eta \neq 0 \) since \( \lambda > 3 \).
Note that as $\theta \to 0$, $V_{11} \to V_{20}$ and $V_{11} \to V_{00}$

so

$$x_{10} = 2q(0) + x_{01} = 2q(\theta)$$

where

$$q(\theta) = 3C(3,3,3) + 2Q(3, V_{11}(\theta)) + 2Q(3, V_{11}(\theta)) + 2Q(3, V_{00}(\theta))$$

so check $q(0) = 3C(3,3,3) + 2Q(3, V_{20}) + 4Q(3, V_{00})$

This is called a lattice function.

This was proven by Sattinger by using the symmetry and L.S. reduction. We have shown it with direct calculation.

$\theta$-dependence arises only from the quadratic term. There also allow one to get squares.

(my proof is much simpler)

Hexagonal Lattice

Here $w(x,t) = \sum_{j=1}^{n} z_j(t) e^{k_j \cdot x}$

Notation: $(z_1, z_2, z_3, z_4, z_5, z_6, z_7) \rightarrow (z_1, z_2, z_3, z_4, z_5, z_6, z_7)$
reflection $(z_1, z_2, z_3, z_4, z_5, z_6) \rightarrow (z_1, z_6, z_5, z_4, z_3, z_2)$
(can do more notation to get rest of equation)
Thus we have all equations. Since we have $F_1$, note that $z_4 = \overline{z_1}$, $z_5 = \overline{z_1}$, $z_6 = \overline{z_3}$

$F_1(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_1, z_2, z_3, z_4, z_5, z_6)$

Also $(x,y) \rightarrow (x,-y)$ implies reflect about y-axis giving $z_1 \rightarrow z_4$, $z_2 \rightarrow z_3$

$z_5 \rightarrow z_6$
\[ F_1(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_1, z_2, z_3, z_4, z_5, z_6) \]
and so on.

Let's recall the following equality:

\[ k_2 + k_6 = k_1 \]
and so on.

**Translation**:

\[ (x_1 + g_1, x_2 + g_2) \rightarrow e^{i_1} g_1 \]
\[ (x_1 + g_1, x_2 + g_2) \rightarrow e^{i_2} g_2 \]
\[ e^{i_1} g_1 + e^{i_2} g_2 \]
\[ = \sum_{n, m, \ell} e^{i_1} g_1 e^{i_2} g_2 = \sum_{n, m, \ell} e^{i_1} g_1 e^{i_2} g_2 \]
\[ = \sum_{n, m, \ell} e^{i_1} g_1 e^{i_2} g_2 \]
\[ = \sum_{n, m, \ell} e^{i_1} g_1 e^{i_2} g_2 \]
\[ = e^{i_1} g_1 e^{i_2} g_2 \]
\[ = \alpha z_1 z_2 z_3 \]

So this means

\[ F_1 = \alpha_1 \beta_1^3 (n - p) + \beta_2 (m - q) + \beta_3 (l - r) \]

Clearly if we choose \( n - p = 1, m - q = 0, l - r = 0 \) we will satisfy this. However, we also have

\[ k_2 + k_6 = k_2 - k_3 = k_1 \]
so we must have

\[ m = 1, r = 1 \]
which gives quadratic term. Thus, up to cubic terms we have

\[ F_1 = \alpha z_1 z_2 z_3 + \beta z_2 z_3 \]

Flipt around x-axis means interchanging \( z_2 \) and \( z_3 \), doesn't change \( F_1 \).

So we get \( \alpha z_2 \rightarrow \alpha z_3 \)

\[ \Phi \text{ invariant under } (z_1, z_2, z_3) \rightarrow (\bar{z}_1, \bar{z}_2, \bar{z}_3) \]
\[ \text{(reflection)} \Rightarrow \Phi(\tilde{z}) = \Phi(\tilde{z}) \]

Thus, all coefficients must be real.

(Note: this is different from reality.)
Thus we get after some manipulation and scaling

\[
\begin{align*}
\frac{d^2 z_1}{dt^2} &= \frac{\beta}{\gamma} \lambda + a \frac{\bar{z}_1}{\bar{z}_3} z_3 - b \frac{z_1^2}{z_2} \bar{z}_2 \left( |z_2|^2 + |z_3|^2 \right) \\
\frac{d^2 z_2}{dt^2} &= \frac{\beta}{\gamma} \lambda + a \frac{\bar{z}_2}{\bar{z}_3} z_3 - b \frac{z_2^2}{z_3} \bar{z}_3 \left( |z_2|^2 + |z_3|^2 \right) \\
\frac{d^2 z_3}{dt^2} &= \frac{\beta}{\gamma} \lambda + a \frac{\bar{z}_3}{\bar{z}_1} z_1 - b \frac{z_3^2}{z_1} \bar{z}_1 \left( |z_2|^2 + |z_3|^2 \right)
\end{align*}
\]

\(a, b, c\) are all real.

What are solutions?

We choose \(a > 0\) w.c.o. (since if \(a < 0\) then change \(z_j \rightarrow -z_j\) & get same back w.t.h.n.a.c.o)

Write \(z_j = r_j e^{i\theta_j}\) & we get

\[
\begin{align*}
\dot{r}_1 &= \lambda r_1 + a r_2 r_3 \cos(\theta_2 - \theta_3 - \theta_1) - b \frac{r_1^2}{r_3} \left( r_1 r_2^2 + r_3 r_2^2 \right) \\
\dot{r}_2 &= \lambda r_2 + a r_1 r_3 \cos(\theta_1 + \theta_2 - \theta_3) - b \frac{r_2^2}{r_3} \left( r_2 r_1^2 + r_3 r_1^2 \right) \\
\dot{r}_3 &= \lambda r_3 + a r_1 r_2 \cos(\theta_2 - \theta_1 - \theta_3) - b \frac{r_3^2}{r_1} \left( r_3 r_1^2 + r_1 r_2^2 \right)
\end{align*}
\]

\[
\begin{align*}
\dot{\theta}_1 &= a \frac{r_2 r_3}{r_1} \sin(\theta_1 - \theta_2 - \theta_3) \\
\dot{\theta}_2 &= a \frac{r_1 r_3}{r_2} \sin(\theta_1 + \theta_2 - \theta_3) \\
\dot{\theta}_3 &= a \frac{r_1 r_2}{r_3} \sin(\theta_2 - \theta_1 - \theta_3)
\end{align*}
\]

If \(a = 0\) so we are left with a bunch of algebraic equations.

\[
\begin{align*}
r_2 r_3 &= 0 + r_1 = \pm \sqrt{\frac{a}{b}}
\end{align*}
\]

Hexagon \(r_1 = r_2 = r_3 = R_0\) where \(\lambda + a R_0 \left( b + 2c \right) R_0^2 = 0\) (solve quadratic.)
Finally, we have rectangles
\[ r_2 = r_3 = r_1 \Rightarrow \]

\[ \begin{align*}
\lambda r_1 + a r_2^2 + b r_2^3 - 2 cr_1 r_2^2 &= 0 \\
\lambda r_2 + a r_2 r_1 - r_2 (b r_2^2 + c r_2 + c r_1^2) &= 0
\end{align*} \]

Solve (2) for \( r_2 \) by dividing by \( r_2 \) first and getting \( r_2 \) as a function of \( r_1 \). Substitute this into (1) to get a cubic. You will find several solutions, but only one we will focus on:

\[ r_1 = -\frac{a}{b - c} \]

\[ r_2 = r_3 = \pm \sqrt{\frac{1}{b + c} \left( \lambda - \frac{a^2}{b - c} \right)} \]

This branch does not hit \( r_1 \) from zero up as \( \lambda \to 0 \).

Note that when \( \lambda = \frac{a^2 b}{b - c} \) \( r_1 \) \( \text{ROLL} = r_1 \text{RECTANGLE} \) and when

\[ \lambda = \frac{(2b + c) a^2}{(b - c)^2} \]

they meet the \text{HEXAGON} branch. Let \( b - c < 0 \).

Then when \( \text{ROLLS} \) \text{HEXAGON} \( \text{arg}(z_1 z_2 z_3) = 0 \)

\[ x = \frac{-a^2}{4(b + c)} \quad \beta = \frac{a^2 b}{(b - c)^2} \]

\[ \varphi = \frac{a^2 (2b + c)}{b - c} \]
Before turning to a stability analysis, we consider a simpler vehicle where \(a = 0\). Our mean no quadratic terms. If \(a = 0\), solving something about \(b + c\) (cf. next square).

\[
\begin{align*}
\dot{r}_1 &= r_1 (\lambda - b r_1 - c (r_1^2 + r_2^2)) \\
\dot{r}_2 &= r_2 (\lambda - b r_2 - c (r_1^2 + r_2^2)) \\
\dot{r}_3 &= r_3 (\lambda - b r_3 - c (r_1^2 + r_2^2))
\end{align*}
\]

Rolls \( r = \sqrt{r_1^2 + r_2^2 + r_3^2} \)

\[
\begin{bmatrix}
\lambda - 3 b r_1^2 & 0 & 0 \\
0 & \lambda - c r_1^2 & 0 \\
0 & 0 & \lambda - c r_1^2
\end{bmatrix} \rightarrow 
\begin{bmatrix}
-2 \lambda & 0 & 0 \\
0 & \lambda (1 - \frac{c}{r_1^2}) & 0 \\
0 & 0 & \lambda (1 - \frac{c}{r_1^2})
\end{bmatrix}
\]

Stable if \( c > 5 \) so \( \lambda \) unsensitive. \( b > c > 0 \)

Hex \( r_1 = r_2 = r_3 = \sqrt{\frac{\lambda}{b r_2 c}} \)

\[
M = \begin{bmatrix}
\lambda - 3 b r_1^2 - 2 c r_1^2 & -2 c r_1^2 & -2 c r_1^2 \\
-2 c r_1^2 & \lambda - 3 b r_1^2 - 2 c r_1^2 & -2 c r_1^2 \\
-2 c r_1^2 & -2 c r_1^2 & \lambda - 3 b r_1^2 - 2 c r_1^2
\end{bmatrix}
\]

Note \( \alpha = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \) has eigenvalues 0, 3d

So \( M = (\lambda - 3 b r_1^2) I - 2 c r_1^2 \) I has eigenvalue

\[
\lambda - 3 b r_1^2, \quad \lambda - 3 b r_1^2 - 6 c r_1^2 = b + 2 c - 3 b - 6 c = -2 (2 b) < 0
\]

So \( \lambda - 3 b r_1^2 - 6 c r_1^2 = b + 2 c - 3 b - 6 c = -2 (2 b) < 0 \), \( c > 5 \)

\[
r_1^2 (b + 2 c - 3 b) = 2 (c - b) r_1^2 \leq 0 \quad c < 5
\]
So in pure cubic case would be stable when $b > c > 0$.

Back to full problem.

\[
\begin{align*}
\lambda - 3br^2 & \quad 0 & \quad 0 & \quad 0 & \quad \lambda = \sqrt{\frac{\lambda}{b}} \\
0 & \quad \lambda - cr^2 & \quad ar & \quad \Rightarrow \lambda = \frac{\lambda}{b} \\
0 & \quad ar & \quad \lambda - cr^2 & \quad \Rightarrow \begin{bmatrix}
-2br^2 & 0 & 0 \\
0 & (b-c)r^2 & ar \\
ar & (b-c)r^2 & 0
\end{bmatrix}
\end{align*}
\]

\[\Rightarrow (b-c) r^2 \pm ar \text{ are eigenvalues. Paticularly,}
\Rightarrow \quad \text{we have}
\Rightarrow \quad \frac{-br^2}{b} \lambda > \frac{a^2}{b} \lambda \Rightarrow \lambda > \frac{a^2}{(b-c)^2} = \beta
\]

Trace $= 2(b-c)r^2 < 0 \Rightarrow b < c$, which is why we drew!! If $b-c > 0$ then rolls never stable.

Oops $\lambda = \mu$.

Hexagon
\[
\begin{bmatrix}
m - 3br^2 - 2cr^2 & -2cr^2 + ar & -2cr^2 + ar \\
-2cr^2 + ar & m - 3br^2 - 2cr^2 & -2cr^2 + ar \\
-2cr^2 + ar & -2cr^2 + ar & m - 3br^2 - 2cr^2
\end{bmatrix}
\]

Writing this as $(m - 3br^2 - ar) I + (ar - 2cr^2) J$.

Using same trick as before, we get eigenvalues:

\[m - 3br^2 - ar, \quad m - 3br^2 - ar + 3(ar - 2cr^2) \text{ (twice)}\]

Using $m - ar - (b + c)r^2 = 0$ we get

\[-2ar - 2(b + c)r^2 < 0 \quad \text{for stability}\]

\[ar - 2(b - 2c)r^2 < 0\]
\( 2ar > -2(b-c)r^2 \quad b > c \text{ always true} \)

\( a > (b-c)r \quad bcc \text{ (drawn)} \)

\( a < 2(b+2c)r, \quad a > (c-b)r \)

\( \Rightarrow (c-b) < 2(b+2c) \quad \text{which is always true if } b > c \)

Note \( r = \frac{1}{2(b+2c)} \left[ a \pm \sqrt{a^2 + 4m(b+2c)} \right] \)

Note * saw \( r > \frac{a}{2(b+2c)} \) so root is unstable

First part of *

we also need * \( r < \frac{a}{c-b} \). But when \( r = \frac{a}{c-b} = \frac{a}{b-c} \)

Thus when hex hits the rectangles so we have proven most of our picture.

Proving short instability of rectangle by again.

When \( b > c \) then condition * is always true and rectangle hex are stable for all \( x \). Rolls are never stable when \( b > c \).

Beyond lattice of c.

Recall in a finite square domain, \( \mathbb{L} \times \mathbb{L} \)

we can have \( h^2 = \frac{4\pi^2}{L^2} [n^2 + m^2] \)

Or \( n^2 + m^2 = \frac{L^2}{4\pi^2} h^2 \)
Treat $\hbar^2$ as fixed from physical parameters & let $L \approx \hbar \gamma$. For example let $L$ be such that $\frac{L}{\gamma \hbar^2} = 1$

Then only possible $(n,m)$ are $(\pm 1,0)$, $(0, \pm 1)$ suppose choose $L$ a bit bigger so that $\frac{L}{\gamma \hbar^2} = 5$, so $n^2+m^2 = 5$, $(\pm 2, 1)$, $(\pm 1, 2)$ and $\pm \infty$.

Here is a picture of what is going on as $L$ grows

Note first intersection core is radius $\sqrt{5}$, with $8$-dimensional unit (space $i(x+ty)$, $j(-x+ty)$, $2e^{i(x+ty)}$, $2e^{i(x-ty)}$, $2e^{i(x+ty)}$, $2e^{-i(x+ty)}$).

HW is to sketch these and derive equations for bifurcation problem radius $25$ has 12-dim.

You derive:

\[
\frac{d^2z_1}{dt^2} = m_1 z_1 + a z_1 \bar{z}_2 - b |z_1|^2 z_1 - b_1 |z_2|^2 z_1
\]

\[
\frac{d^2z_2}{dt^2} = m_2 z_2 + c z_2 - d_1 |z_2|^2 z_2 - d_2 |z_1|^2 z_2
\]
WLOG let \( a = 1 \) \(, c = \pm 1 \) (rescaling amplitude, but cannot fix \( s(\mu_0 + c) \) all par anrea), let \( z_1 = Re^i\phi \) \(, z_2 = Se^{i\theta} \), let \( \Theta = \gamma - 2\phi \)

\[
\begin{align*}
\mu' &= \mu R + RS \cos \Theta - b_1 R^2 - b_2 S^2 R \\
S' &= \mu S + R^2 \cos \Theta - d_1 S^3 - d_2 S^2 R^2 \\
\Theta' = \left( -\frac{R^2}{S} - 2S \right) \sin \Theta \\
X &= s \cos \theta \, , \, Y = s \sin \theta \, , \, Z = R^2 \\
X' &= \mu_2 X \pm Z + 2Y^2 - d_1 X(X^2 + Y^2) - d_2 XZ \\
Y' &= \mu_2 Y - 2XY - d_1 Y(X^2 + Y^2) - d_2 YZ \\
Z' &= 2Z(M_1 + X - b_2 Z - b_2(X^2 + Y^2)) \\

(1) \text{ Pure mode } \quad Y = Z = 0 \quad X^2 = \frac{m_2}{d_1} \quad (n = 0, \theta = 0, Z = 0, \frac{m_2}{d_1}) \\
X \neq \Theta \quad (Note \ Z = 0 \text{ is not in varint all!})

(2) \text{ Mixed mode } Y = 0 \quad (\theta = 0, \pi) \\
M_1 + X - b_2 Z - b_2 X^2 = 0 \quad m_2 X \pm Z - d_1 X^3 - d_2 XZ \\

(3) \text{ Really cool! } \quad c = -1, \text{ travelling waves} \\
\Xi = \frac{\partial X}{\partial \phi} \\
Note \ R \frac{d\phi}{dt} = RS \sin(\gamma - 2\phi) \\
= RS \sin \Theta

If \( \theta \neq 0, \pi \) then must have \( \frac{R^2}{S} - 2S = 0 \)
\[
\Rightarrow \text{ take } c = -1 \Rightarrow R^2 = 2S^2 \\
\Rightarrow R = \sqrt{2S}
\]

\[
0 = M_1 + \mu_2 S \cos \theta - 2b_1 S^2 \cos \theta - b_2 S^2 \\
0 = \mu_2 S^2 - 2S \cos \theta - d_1 S^3 - 2d_2 S^2
\]
Multiply first by 2 and then solve

\[2u_1 + u_2 - (4b_1 + 2b_2)S^2 - d_1 S^2 - 2d_2 S^2 = 0\]

\[
(2u_1 + u_2)/(4b_1 + 2b_2 + d_1 + 2d_2) = S^2
\]

\[
\cos \theta = (2b_1 + b_2)/(2u_1 + u_2)
\]

\[
\cos \theta = -\frac{d_1 (2b_1 + b_2) S^2}{S} \text{ if } |u_1| < 1
\]

Prove \( \theta \) and thus determine the wave form since

\[
\frac{d\theta}{dt} = 2S \omega \sin \theta
\]

Since \( \theta \neq 0, 1, 7 \) and \( S \neq 0 \), \( \frac{d\theta}{dt} = \omega \neq 0 \) and \( \theta = \omega t \)

\[
\Rightarrow \ u(x, t) = \sqrt{2} S e^{i(x + \omega t + \phi_0)} + S e^{i(x + \omega t + \phi_0)} + \phi_0
\]

\( \phi_0 - \phi_0 = 0 \) Pretty bizarre.

Called Drift Instability

There are many other modes interacting. For example if two wave modes are \( n, n+1 \) with \( n \neq 0 \) then it is back to our standard friends without resonant term e.g. (2, 1, 3) cannot get

\[2 = n_2 + m_3 \text{ except } n_1 = 1, m_2 = 0 \]

or \( n_2 = 2, m_2 = 2 \). But this is not cubic.
So far we have spent all our time on square lattice domain. What about radially symmetric solutions.

For example, let's consider a reaction diffusion equation in polar coordinates:
\[
\frac{\partial u}{\partial t} = F(u,v) + \nabla \left[ \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} + \frac{u}{r^2} \right]
\]

Can we get bifurcation to radially symmetric solutions? (These are like 1-d solutions)

\[
\frac{\partial V}{\partial t} = \nabla \cdot \left[ \frac{\partial V}{\partial r} + \frac{1}{r} \frac{\partial V}{\partial r} \right]
\]

Linearized eigenvalue \[
\frac{\partial^2 V}{\partial r^2} + \frac{\partial V}{\partial r} + r^2 h^2 V = 0
\]

Look at eigenvalue of \[
\frac{\partial^2 y}{\partial r^2} + \frac{\partial y}{\partial r} + r^2 h^2 y = 0
\]

Let \( r h = x = \) \[
X^2 \frac{\partial^2 y}{\partial x^2} + X \frac{\partial y}{\partial x} + X^2 y = 0
\]

Solution is \( J_0(x) \) Bessel Function

So \( V(r,t) = e^{J_0(kr)} \frac{1}{r^2} \) where \( V \frac{\partial^2}{\partial r^2} = A - h^2 I \) looks familiar!