Some special cases:

Cyclic systems \( F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \)

\[
  \dot{x}_i = F(x_i, \sum_{j=0}^{N-1} A_{i-j} x_{i-j}) \quad i = 0, \ldots, N-1, \quad \text{mod } N
\]

\( A_{i-j} \) are non-negative

Note that you could regard this geometry as a torus, and your interaction with unit \( k \) depends only on \( k-1 \) your "distance" from \( i \).

These are often used in biology as simplification of neural geometry.

They are homogeneous in sense that interactions depend on difference of location (index) and not on absolute location.

Let \( \overline{x} \) solve

\[
  \overline{x}(0) = F(\overline{x}, \sum_{j=0}^{N-1} A_{i-j})
\]

Linearize (let just forget first term here)
\[ B = F_1(\hat{X}, \hat{A}) \]

\[ y_j = b y_j + \sum_{j=0}^{n-1} c_j y_{j-1}, \quad c_j = F_2(\hat{x}, \hat{A}, \lambda_j) \]

**Claim** \[ y_1 = e \cdot e^{\frac{\pi i j}{n-1}} \] is a solution.

\[ \lambda e^{\frac{\pi i j}{n-1}} = B e^{\frac{\pi i j}{n-1}} + \sum_{j=0}^{n-1} c_j e^{-\frac{\pi i j}{n-1}} \]

\[ \lambda = B e^{\frac{\pi i j}{n-1}} + \sum_{j=0}^{n-1} c_j e^{-\frac{\pi i j}{n-1}} \]

**For each** \( e = 0, 1, \ldots, n-1 \), \( D_{11} \) is an \( n \times n \) system so get \( N \) \( n \times n \) instead of \( 1 \) \((nN \times nN)\) !

\[ \sum_{j=0}^{n-1} c_j e^{-\frac{\pi i j}{n-1}} \] is the discrete Fourier Transform of \( c_j \)

**Key idea** as before: Need to use homogeneity + eigen properties of \( A \). adjacency matrix.

**0D example**: 20 lattice walks

- periodic B.C.s
- nearest neighbors on a line
- etc etc

**Let applying to a ring or Bravais lattice** with \( N \) \( N \) coupling.
\[ y_i = f(x_i, y_i) + D_x (x_i-1 - 2x_i + x_{i+1}) \quad i = 0, \ldots, N-1 \]
\[ y_i = g(y_i, y_i) + D_y (y_i-1 - 2y_i + y_{i+1}) \]

Linearize:

\[
\begin{bmatrix}
  u_i \\
  v_i
\end{bmatrix} =
\begin{bmatrix}
  A & B \\
  0 & D_y
\end{bmatrix}
\begin{bmatrix}
  u_{i+1} - 2u_i + u_{i-1} \\
  v_{i+1} - 2v_i + v_{i-1}
\end{bmatrix}
\]

\[
e^{\frac{2\pi i}{N}} = 2 \cos \frac{2\pi l}{N} \quad l = \text{cyclic sum over } \{0, \ldots, N-1\}
\]

(Note: For \( N = 2 \), get \( 0, -4 \), which is what we had before, but we will actually get \( 0, -2 \) and \( \text{we have counted twice} \) since \( i-1 = i+1 \))

Letting \( A = \begin{pmatrix} a & b \\ \bar{a} & \bar{b} \end{pmatrix} \)

We have a series of matrices:

\[
M_0 = \begin{pmatrix} a + D_x y_0 & B \\ \bar{B} & \bar{a} + D_y y_0 \end{pmatrix}
\]

\[
\text{Tr } M_0 \geq \text{Tr } M \quad \forall \text{ since } M \leq M_0
\]

\[
\det M = \text{det } M_0 + D_x D_y M
\]

\[
+ M \begin{pmatrix} 2D_y + 6D_x \end{pmatrix}
\]

For \( N = 1 \) and small \( M_z^2 - \frac{4D_y^2}{N^2} \)
Try some sims for Missyken!

ample2  Global inhibition

A classic example combines local interactions with global negative feedback:

\[ x_i = -x_i + f(\frac{\alpha (x_{i+1} + x_{i+1} + \cdots + x_{i+N}) - \beta}{N}) \quad \alpha, \beta > 0 \]

Linearize about equilibrium, \( x_i = 0 \):

\[ y_i = -y_i + a(y_{i+1} + y_{i+1} + \cdots + y_{i+N}) - b y_j \]

\[ y_i = e^{\lambda t} \frac{2 \pi \sqrt{N}}{N} \quad a, b > 0 \]

\[ x_0 = -1 + 3a - Nb \quad \lambda = -1 + a \left( 1 + 2 \cos \frac{1}{N} \right) \]

Note that \( 3a > a \left( 1 + 2 \cos \frac{1}{N} \right) \) if \( a > 0 \)

but \(-Nb\) can dominate making \( \lambda_0 < 0 \)

\( \lambda_1 = \lambda_{N-1} \) is larger than all others.

Thus the most unstable mode will be

\( \lambda = \lambda_{N-1} \) and the pattern will always be one full wavelength, no matter what the initial size \( (\text{size, size, size}) \).

Thus, a mechanism to get size invariant pattern.
The point is that what we have here is a negative interaction between negative and positive feedback. As an example, you should consider instead:

$$-\beta \sum_{j=-m}^{m} X_{i+j}$$

The dominant eigenvalue depends on $m$. Need to sum

$$\sum_{j=-m}^{m} e^{-\lambda_j}$$

which you should be able to do.

We will get to nonlinear systems and bifurcation shortly but want to turn to some known continuous space examples. In the second half of the course, we will examine many other examples of pattern formation.

Let first consider a general reaction diffusion equation in $\mathbb{R}^n$ on a one-dimensional domain of length $L$ with different boundary conditions.
We will just describe RD equation

\[
\begin{align*}
\frac{\partial u}{\partial t} &= f(u,v) + D_u \frac{\partial^2 u}{\partial x^2}, \\
\frac{\partial v}{\partial t} &= g(u,v) + D_v \frac{\partial^2 v}{\partial x^2} \\
\end{align*}
\]

BCS \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \quad a \leq x \leq L

\begin{align*}
\text{Neumann or} & \quad \text{No flux BCS} \\
\text{Periodic} & \quad u(0,t) = u(L,t) \\
\text{Dirichlet} & \quad v(0,t) = v(L,t) \\
\end{align*}

We assume that \( f(\bar{u},\bar{v}) = g(\bar{u},\bar{v}) = 0 \)

Linear stability theory.

\[ u(x,t) = \bar{u} + w(x,t); \quad v(x,t) = \bar{v} + \tilde{z}(x,t) \]

Let \( \tilde{z} = (f \gamma) \quad dF | \bar{u},\bar{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = M_0 \)

Assume \( \text{det } M_0 > 0 \); \quadTv \quad M_0 < 0

\[ \begin{bmatrix} w \\ \tilde{z} \end{bmatrix}_t = \begin{bmatrix} M_1 \end{bmatrix} \begin{bmatrix} w \\ \tilde{z} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \tilde{z} \end{bmatrix} \]

Let's look at three different linear eigenvalue problems

\begin{enumerate}
\item \( \lambda \hat{y} = \hat{y}_{xx} \quad \hat{y}(0) = \hat{y}(L) = 0 \)
\item \( \lambda \hat{y} = \hat{y}_{xx} \quad \hat{y}_x(0) = \hat{y}_x(L) = 0 \)
\item \( \lambda \hat{y} = \hat{y}_{xx} \quad \hat{y}(0) = \hat{y}(L) \hat{y}_x(0) = \hat{y}_x(L) \)
\end{enumerate}
Standard 2D wave: $\alpha, \beta \in \mathbb{R}$

\begin{align*}
(\theta) \quad \phi_n(x) = \sin \frac{n\pi x}{L} \\
(\rho) \quad \phi_n(x) = \sin \frac{n\pi x}{L}, \quad \cos \frac{n\pi x}{L} \quad \lambda_n = -\frac{4\alpha^2}{L^2}
\end{align*}

Thus we can use this idea to solve the linear equation \((\theta)\) under the three assumptions on BCS.

\begin{align*}
(W) &= e^{i t} \phi_n(x) [p] \quad \text{with} \\
Y[p] &= M \delta [p] + \lambda_n \begin{bmatrix} 0 & 0 \\ 0 & \partial_y \end{bmatrix} [p]
\end{align*}

Thus we get eigenvalues $\lambda_n$ and plot these as a function of $n$. Write $Re \lambda_n = \alpha_n$

called "dispersion relation" or wavenumber.

Want some maximum at $n = 0$. Note $\alpha_0 = 0$ symmetry due to isotropic diffusion (diffusion jump in all direction).

These ideas work for any domain---key is to know the eigenfunctions for the particular domain.
For any kind of system of equations
with a nice linearization and regular enough domain, we can find eigenvalues for the linear equations by using a change of variable transformation

One class of models we will look at later are integral operators.

A crucial property we will want is some sort of linearity. Otherwise, we cannot get eigenvalues analytically.

Let take as an example convolution equation:

$$\lambda u(x) = \int_0^1 k(x-y)u(y)\,dy \quad x \in [0,1]$$

where we suppose that $k(x+1) = k(x)$ so $k$ is a periodic "kernel".

The eigenvalues of this operator are easy to find. Try

$$\lambda u(x) = e^{2\pi i nx}$$

$$\lambda u(x) = ne^{2\pi in} = \int_0^1 k(x-y)e^{2\pi iny}\,dy$$

Let $x = x-y$ so $dx = -dy$ and we get

$$\lambda e^{2\pi inx} = \int_0^1 k(x')e^{2\pi inx}\,dx'$$

$$\Rightarrow \lambda = \int_0^1 k(x')e^{2\pi inx}\,dx'$$
For my reason we can study many non local problems. In 2 dimensions typically consider infinite periodic domain or infinite domain.

Infinite domain have some mathematical problems, due to a continuum of joints consider, say

\[ \Delta u = \lambda u \quad u : \mathbb{R}^2 \rightarrow \mathbb{R} \]

Bounded function are \( u(x) = e \)

and \( \lambda = -k^2 \)

There are infinitely many functions for each \( \lambda \).

One simplification that is sometime done is restrict there to a periodic lattice such as square or hexagonal lattice.

We will look at this type of system later on.

Consider a square domain in \( [0, L] \times [0, L] \) with

\[ \Delta u = \lambda u \quad u : \mathbb{R}^2 \rightarrow \mathbb{R} \]

and assume periodic boundary condition.

Then \( u(x, y) = e^{\frac{-2\pi i (nx + my)}{L}} \) with \( \lambda = -\frac{4\pi^2}{L^2} (n^2 + m^2) \)
So far, for a given \( \lambda \), there could be 8 or 5 as little as four dimensional eigenpace

\((n_0, m_0) (n_0, -m_0) (n_0, m_0) (n_0, -m_0)\)

or much larger. Say \( \lambda = \frac{-4\pi^2}{L^2} \). 25

\(n^2 + m^2 = 25\)

\((\pm 5, 0), (0, \pm 5), (\pm 3, 4), (\pm 4, 3)\)

\((\pm 3, -4), (\pm 4, -3)\)

12 dimension.!!

Summary. For continuous & discrete smalup problem that have some homogeneity or symmetry properties, we can usually find eigenvalues & obtain the decomposition problem into simpler one.

Two more examples:

(1) \( u_{t t} = u + F_1 \left( k_{11} x u_t - k_{12} x u_x \right) \) \hspace{1cm} \text{Assume } k_{ij} + I = \delta_{ij}\text{ is ind of } x. \hspace{1cm} \text{(or } F(0) = 0\text{)}

\( u_{t t} = u + F_2 \left( k_{21} u_t - k_{22} u_x \right) \)

Linearize about homogeneous state \hspace{1cm} \text{so } u \equiv 0\text{.}

\( u_{t t} = u_1 + \beta_1 \left( k_{11} x u_t - k_{12} x u_x \right) \)

\( u_{t t} = u_2 + \beta_2 \left( k_{21} u_t - k_{22} u_x \right) \)

Problem: \( k_{ij} \times u = \lambda u \) may not have same eigenfunction. So, no simplification is possible. Need some eigenfunction formula.
Example 2

$$\vec{u}_1 = f(x) \hat{x} + B \hat{y} \Delta \vec{u}$$

with boundary condition

must choose BC's so that eigenfunctions of $$\Delta w = \lambda w$$ are same!

Pen can simplify. (For example could not have $$\nabla \cdot \vec{u}, \lambda = 0$$
and $$u_2 = 0$$ on $$\partial \Omega$$ for $$\vec{u} = (u_1, u_2)$$)

In practice, this is not a bad restriction, but you should be aware of it.

Now what can we do with all this?

It is time to turn to nonlinear equations and bifurcation theory, which I
will review for you.

Most of the ideas we will use will be applied to spatial problems, but it
is easier to start with ODEs and then assert that the same holds in
infinite dimensions (it usually does!)
\[ \frac{d\mathbf{x}}{dt} = f(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^n \]

\( \mathbf{x} \) is a fixed point or equilibrium if \( \mathbf{x} \) is a solution of \( f(\mathbf{x}) = 0 \).

Example \( \dot{x} = x - x^3 \), \( x = 0, \pm 1 \)

\( \dot{x} = x(1-x^2-y^2)-y \quad \dot{y} = y(1-x^2-y^2) + x \)

has \( (0,0) \) as fixed point.

Let \( \mathbf{A} = Df = \frac{\partial f}{\partial \mathbf{x}} \) be the linearization.

If all eigenvalues of \( \mathbf{A} \) have all negative real parts, then \( \mathbf{x} \) is asymptotically stable.

If at least one eigenvalue has a positive real part, \( \mathbf{x} \) is unstable.

If no eigenvalue of \( \mathbf{A} \) have zero real part, we say that \( \mathbf{x} \) is a hyperbolic equilibrium. The behavior near \( \mathbf{x} \) is the same as the linear equation.

Center manifold theorem provides a way to study nonlinear systems near the non-hyperbolic equilibria.

We write \( \frac{d\mathbf{x}}{dt} = f(\mathbf{x}, \mathbf{u}) \) \( \mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m \)

near non-hyperbolic equilibrium as

\[ \frac{d\mathbf{y}}{dt} = \mathbf{A} \mathbf{y} + \mathbf{g}(\mathbf{y}) \]
\[
\frac{dy}{dt} = B_y + g_y(y, z, \frac{dz}{dt})
\]

\[
\frac{dz}{dt} = C_z + g_z(y, z, \frac{dz}{dt})
\]

Center manifold theorem guarantees existence of this manifold tangent to \( z = 0 \) at \( z = 0 \)

It is invariant (start here, stay here)

Locally \( z = h(y) \) so that

\[
\frac{dy}{dt} = B_y + g_y(y, h(y))
\]

Note: \( \text{dim } y = n \) typically much less than \( \text{dim } x = n \) so this is much simpler.

If here are parameters, \( m \) we can write

\[
\frac{d\lambda}{dt} = 0 \quad \frac{dy}{dt} = B_y + g_y(y, z, m)
\]

\[
\frac{dz}{dt} = C_z + g_z(y, z, m)
\]

\[
\Rightarrow \quad \frac{dy}{dt} = B_y + g_y(y, h(y, m))
\]
You will learn all of this in Rubin's class. But let's do an example.

$$\frac{dy}{dt} = B_0 + B_1 y + \cdots$$

$$\frac{dz}{dt} = C_0 + C_1 y + \cdots$$

Write \( z = h(y) \)

$$\frac{dz}{dt} = \frac{dh}{dy} \frac{dy}{dt}$$

\[ \frac{dh}{dy} \left[ B_0 + B_1 y + \cdots \right] = C_0 + C_1 y + \cdots \]

Need to solve \( \frac{dh}{dy} \) for \( h(y) \), \( y_0(y) \).

PDE: Only need it near \( y = 0 \), so usually expand in polynomial to match coefficients.

\( \frac{dx}{dt} = ax - xy \), \( \frac{dy}{dt} = -y + x^2 \), \( \frac{du}{dt} = 0 \)

(\( x = y = 0 \)) equilibrium point \( \lambda = -1 \), \( u \)

For \( m < 0 \) stable, \( m > 0 \) unstable \( m = 0 \) non-hyperbolic.

So \( y = h(x_0) \), \( \frac{dy}{dt} = \frac{dh}{dx} \frac{dx}{dt} + \frac{dh}{du} \frac{du}{dt} \)

\( \frac{dy}{dt} = -h(x_0)^2 + x_0 = \frac{dh}{dx} [mx - h(x)] \)

Write \( h(x_0) \) as polynomial.

\( h = ax^2 + bx + c + \cdots \)

\( x^2 - [ax^2 + bx + c + \cdots] = (2ax + bx) x [m-ax^2-bx-c+\cdots] \)

\( a = 0 \), \( x^2 \), \( x^3 \), \( x^4 \), \( b = 0 \), \( m \), \( c = 0 \)

So \( y = x^2 + \cdots + \frac{1}{2} \frac{dx}{dt} [mx - x^2] \).
This method is great for simple examples, but really suffers for realistic problems. So we'll use the method of multiple scales and perturbation theory later on!

\[ \frac{dx}{dt} = mx \pm x^3 \quad \frac{dx}{dt} = mx + x^2 \]

These are the three types of steady-state bifurcations you get in simple ODEs with eigenvalues. The "problem" with symmetry is that there are usually unstable eigenvalues as we saw in the last few pages.

Before continuing, I want to introduce another way to compute the nonlinear portion of a bifurcation calculation.

It is called the Lyapunov–Schmidt technique and it applies with many functional equations as well. The advantage of LS is that there is no need to first transform the problem into Jordan form (for the linear part).
Let \( F : B \times \mathbb{R} \to B \) be a nonlinear mapping of some function space (or could be \( \mathbb{R}^n \)) but more generally, it is infinite dimensional (like a Banach space or something).

Want solution \( F(u, \lambda) = 0 \) \( (\lambda \in \mathbb{R} \) is parameter \( u \in B \) \). Suppose \( u = 0 \) is a solution for \( \lambda = 0 \).

Linearize \( L(\lambda) = D_u F(0, \lambda) \).

If \( L(\lambda) \) is invertible then from \( L(\lambda)u = 0 \) \( \Rightarrow F(u(\lambda), \lambda) = 0 \) + \( u(\lambda) = 0 \).

This gives a unique solution near \( u = 0 \). Since \( F(0, \lambda) = 0 \) for \( \lambda \) in some neighborhood of \( 0 \), \( u = 0 \) is only small solution.

So if we want to find branches of nontrivial solutions near \( u = 0 \), we had better not have \( L(\lambda) \) invertible. So we assume that \( L(\lambda) \equiv L(\lambda_0) \) has a finite dimension null space.

Assume \( \mathrm{Null} \ L_0 \) is finite dimensional closed subspace of \( B \). \( \mathrm{Null} \ L_0 = \{ f \in B \mid \exists u, L(u) = f \} \) has finite codimension \(( \mathrm{Null} \ L_0 \) is finite dimensional) and \( \dim N_0 = \dim \mathbb{R}^\perp \). (This means \( F \) is held in alternative holds: \( L(u) = f \) if and only if \( N_0 \).)

Define projections \( P : B \to \mathbb{R}_0 \), \( Q : B \to N_0 \).
Write \( u \in B \) as \( u = v + w \) where \( v \in N_{lo} \) and \( w \in N_{lo}^\perp \). Then \( u = Q \eta, \; w = (I-Q)u \).

\[ F(u, \lambda) = 0 \quad \Rightarrow \quad (\lambda P) F(v + w, \lambda) = 0, \quad (I - \lambda P) F(v + w, \lambda) = 0. \]

Regard \((\lambda)\) as a map from \((E-Q)B \to R_{lo}^\perp\) with \( V, \lambda \) fixed. Clearly \( P F(0, \lambda) = 0 \) and \( P L_0 : Du P F(0, \lambda) \) is invertible as a map from \((E-Q)B \to R_{lo}^\perp\) since we have projected out the null space! Thus from \( LFT \) we can uniquely solve for \( w = \tilde{w}(V, \lambda) \) when \( |V| \) and \( |\lambda - \varrho| \) are small. So

\[ \tilde{w}(0, \lambda) = 0. \]

It remains to solve \( F(V, \lambda) = (I-P) F(\tilde{w}(V, \lambda), V, \lambda) = 0 \).

But now \( f \) is a map from a finite dimensional \( N_{lo} \) to another FDS \((E-Q)B\). If we write

\[ V = \sum_{i=1}^n z_i \varphi_i, \quad z_i \in \mathbb{C} \text{ where } \{\varphi_i\}_i \text{ is an} \]

basis for \( N_{lo} \) then we get \( \tilde{E} \tilde{f} = 0 \) span \((E-Q)B\).

Thus solving \((\#)\) is equivalent to solving

\[ f_1(z_1, \ldots, z_n, \lambda) = 0, \quad f_2(z_1, \ldots, z_n, \lambda) = 0, \ldots f_n(z_1, \ldots, z_n, \lambda) = 0, \]

where \( F(\sum_{i=1}^n z_i \varphi_i, \lambda) = \sum_{j=1}^n f_j(z_1, \ldots, z_n, \lambda) \varphi_i \).
How does this help? For those of you who have done any restoration theory, this is all just tautology to the application of the Fredholm Alternative Theorem.

We now apply this to an example problem.
We now apply this to an example from previous lectures. The symmetric coupled path. Let \( F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) and consider:
\[
\dot{X}_1 = F(X_1, X_2, \lambda), \quad \dot{X}_2 = F(X_2, X_1, \lambda)
\]
with \( \lambda \) a parameter.

We assume \( F(\lambda, \lambda, \lambda) = 0 \), with \( \lambda \to 0 \).

Let \( \lambda_0 = 0 \).

Let \( X_1(0) = \lambda(0) + Y_1(\lambda) \) so that we get
\[
\dot{Y}_1 = F(Y_1, Y_2, \lambda), \quad \dot{Y}_2 = F(Y_2, Y_1, \lambda) \quad \text{and} \quad Y_3 = 0 \]
a solution for \( \lambda \to 0 \).

Let \( \dot{A} = \partial F_2(0, 0, \lambda), \quad B = \partial F_2(0, 0, \lambda) \)
be matrices. So full linearization on \( \lambda \)
\[
\begin{bmatrix}
A & B \\
B & 0
\end{bmatrix}
\]
eigenvalues are \( \lambda \) and \( \lambda + A + B \).

By hypothesis \( F(\lambda, \lambda, \lambda) \) there are no zero eigenvalues of \( A + B \) since \( \mathbb{R}^n \)

(1) Assume \( A + B \) has all eigenvalues with negative real parts near \( \lambda = 0 \).

(2) Assume \( A - B \) has zero eigenvalues at \( \lambda = 0 \) and all other eigenvalues have negative real parts near \( \lambda = 0 \).

(3) Let \( \mu(\lambda) \) be the eigenvalue of \( \mu(\lambda) = 0 \) near \( \lambda = 0 \).

We will now derive the equation for the dynamics near \( \lambda = 0 \), using perturbation theory.
Acids: Fred's mum is very hot. Lu = f has a soln
1 + \langle V_j, \tau \rangle = 0 \quad \text{for all } V_j \quad \text{st } L^* V_r = 0.

\langle L^* V, \tau \rangle = \langle V, L\tau \rangle \quad \text{on } s \quad \text{by } L^*

- We had multi-scale: A_{ss} = \text{some function of } t \quad \text{depends on } \frac{\partial s}{\partial t}, \frac{\partial^2 s}{\partial t^2}, ... \quad \text{so } \frac{\partial s}{\partial t} = \frac{\partial}{\partial s} + \text{other terms}.

Let us look for soln which to $s = t$ at rest time
$\dot{y}_1 = F(y_1, y_2, \lambda), \quad \dot{y}_2 = F(y_2, y_1, \lambda) \quad \text{Fast + 14}

\underline{Preliminary - General Taylor series.}

\[ F(y_1, y_2) = A y_1 + B y_2 + Q_1(y_1, y_2) + Q_2(y_1, y_2) + Q_3(y_1, y_2) + \cdots \]

$Q(W, Z) = Q(Z, W)$ is bilinear form.

$Q(a w_1 + b w_2, z) = a Q(w_1, z) + b Q(w_2, z)$ is bilinear form.

$C(W, Z, P)$ is trilinear form.

$\dot{y} = C(W, Z, \lambda)$ and choose $\lambda$ small otherwise.

We may also depend on time and $\lambda$ in above time scales.

Write $y = \sum_{j=1}^{\infty} \varepsilon^j y_j \quad \lambda = \sum_{j=1}^{\infty} \varepsilon^j \lambda_j$.
Let \((A_0^T B_0) = A(0) + B(0)\). Let

\[
\begin{align*}
(A(\lambda), B(\lambda)) &= A_0 \Theta_0 + \lambda (A_1 \Theta_1 + B_1) + \lambda^2 (A_2 \Theta_2 + B_2), \quad \ldots
\end{align*}
\]

Let \((A_0 \Theta_0)^T V = 0\) and \((A_0^T B_0) \, V^* = 0\) with

\[
V^* \cdot V = 1 \quad (WLOG)
\]

\[
\frac{dV}{d\lambda} = \frac{2}{\lambda} \frac{dV}{d\lambda} + \frac{2}{\lambda} \frac{dV}{d\lambda} + \frac{2}{\lambda^3} \frac{dV}{d\lambda} + \frac{2}{\lambda^3} \frac{dV}{d\lambda} + \frac{2}{\lambda^3} \frac{dV}{d\lambda} + \ldots
\]

The way to order \(\lambda^2\) is

\[
\frac{dV}{d\lambda} \equiv \begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix} \frac{dV}{d\lambda}
\]

only nonzero solution is \(p_1 = \begin{pmatrix} -V \\ V \end{pmatrix}\) where \(V\) is nonsingular and function of \(E_1, E_2\) only.

Order \(\lambda^2\)

\[
\frac{dV}{d\lambda} \Rightarrow \begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix} \frac{dV}{d\lambda} + \lambda \frac{dV}{d\lambda} + \lambda^2 \frac{dV}{d\lambda} + \ldots
\]

Write this as

\[
\begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix} \frac{dV}{d\lambda} + \lambda \frac{dV}{d\lambda} + \lambda^2 \frac{dV}{d\lambda}
\]

where

\[
\begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix} \quad \text{has 1 dim nullspace spanned by} \quad \begin{pmatrix} -V \\ V \end{pmatrix}
\]

and

\[
\begin{pmatrix} A_0^T B_0 \\ B_0^T A_0 \end{pmatrix} \quad \text{has 1 dim nullspace} \quad \begin{pmatrix} V^* \\ -V^* \end{pmatrix}
\]

so from PA \(\begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix}\) has 2 dim nullspace \(\begin{pmatrix} -V^* \\ V^* \end{pmatrix}\) \(\Rightarrow\)
\[
-\frac{dr}{dt} + \lambda r \left( \frac{v^*}{v^* \left( A_1 - B_0 \right)} \right) = 0
\]

Claim: \( \eta = 2 \left( A_1 - B_0 \right)^* \). Let's suppose \( \eta > 0 \)

Then \( \frac{dr}{dt} = \lambda r \), so that we have \( r = e^{\lambda t} r(0) \)

does not grow exponentially unless \( \lambda \neq 0 \) (either \( r \to 0 \) or \( r \to \infty \), neither is very good!) so we pick \( \lambda = 0 \) and \( \frac{dr}{dt} = 0 \) so \( r \) is a constant.

Need to prove \( \eta \neq 0 \).

Let \( z(\lambda) \) be eigenvector corresponding to \( \mu(\lambda) \)

with 
\[
\left[ A_1 - B_0 \right] z(\lambda) = \mu(\lambda) z(\lambda)
\]

set \( \lambda = \frac{\partial}{\partial \lambda} \mid_{\lambda=0} \) by det and \( z(0) = \frac{dy}{dx} \mid_{x=0} \)

Multiply both sides by \( v^* \) and take inner product:

\( v^* \cdot (A_1 - B_0) z = 2 \mu(0) \) since \( v^* \cdot v = \frac{1}{2} \)

\( \Rightarrow 2 v^* \cdot (A_1 - B_0) z = \mu(0) > 0 \) by hypothesis.

(Write \( v^* \cdot (A_1 - B_0) z = (A_1 - B_0) v^* \cdot z \), so \( z = 0 \).)

5. we find \( r \) and of \( \epsilon \epsilon \) and \( \lambda = 0 \)

Thus:
\[
\begin{bmatrix}
\beta_0 & \beta_0
\end{bmatrix}
\begin{bmatrix}
\beta_0 \\
\beta_0
\end{bmatrix}
= r^2 \begin{bmatrix}
\bar{\alpha}
\end{bmatrix}
\]
and from this we see that \( \rho_2 = L^\frac{q}{2} \).

Where \(- (A_0 + B_0) \tilde{q} = \tilde{Q} \), since \( A_0 + B_0 \) is invertible we can find \( \tilde{q} \)

\[ \tilde{q} = - (A_0 + B_0)^{-1} \tilde{Q} \]

Summarizing, so far:

\[ \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \varepsilon r \begin{bmatrix} \dot{v} \\ \ddot{v} \end{bmatrix} + \varepsilon^2 r^2 \begin{bmatrix} \dddot{q} \\ q \end{bmatrix} \]

Now we go to cubic order \( \varepsilon^3 \):

\[- (A_0 + B_0) \rho_3 \mathbb{I} = \begin{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} + \varepsilon \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \left[ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right] \end{bmatrix} + [2Q_1(v, v, v) - 2Q_3(v, v, v)] \varepsilon r^3 \\
- [2Q_1(v, v, v) + 2Q_3(v, v, v)] \varepsilon r^3 \]

\[= \begin{bmatrix} C_1(v, v, v) \dot{C}_2(v, v, v) + C_3(v, v, v) - C_4(v, v, v) \\ -C_1(v, v, v) + C_2(v, v, v) - C_3(v, v, v) + C_4(v, v, v) \end{bmatrix} \varepsilon r^3 \]

Applying \( \tilde{q} = A \) to \( \rho_3 \) we get:

\[ \frac{dr}{dr} = \lambda_1 r M_3(0) + \gamma_3 r^3 \]

\[ \gamma_3 = 2 \dot{v}_1 \dot{z} = 2 \left( \dot{v}_1, [2Q_1(v, v) - 2Q_3(v, v) + C_1 - C_2 + C_4] \right) \]

Last hypothesis \( \gamma_3 \neq 0 \).