General 2-unit problem

Let \( x_j \in \mathbb{R}^n \), \( f_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \)

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*}
\]

Want same eqns for \( x_1 \leftrightarrow x_2 \)

\[
\begin{align*}
\dot{x}_2 &= f_1(x_2, x_1) \\
\dot{x}_1 &= f_2(x_2, x_1)
\end{align*}
\]

\[\iff f_1(x_2, x_1) = f_2(x_1, x_2)\]

So:

\[
\begin{align*}
\dot{x}_1 &= f(x_1, x_2) \\
\dot{x}_2 &= f(x_2, x_1)
\end{align*}
\]

Let \( f(u, u) = 0 \) 

Let

\[
\begin{align*}
A &= \partial_1 f(u, u) \\
B &= \partial_2 f(u, u)
\end{align*}
\]

\[
\begin{align*}
y_1 &= Ay_1 + By_2 \\
y_2 &= Ay_2 + By_1
\end{align*}
\]

To see symmetry:

Let \( y_3 = y_1 - y_2 \), \( y_4 = y_1 + y_2 \)

\[
\begin{align*}
y_3' &= Ay_3 - Ay_2 + B(y_2 - y_1) \\
&= (A-B)y_3 \\
y_4' &= (A+B)y_4
\end{align*}
\]

\[\text{so reduce to } 2 \times \text{nn instead of } 2 \times 2 \times \text{nn}!\]
This need only look at spectrum of smaller system.

Example: The Brusselator chemical reaction

This is the canonical reaction for oscillations & pattern formation.

\[ A \rightarrow X \quad B + X \rightarrow Y \quad 2X + Y \rightarrow 3X \]

\[ X \rightarrow E \quad \text{all have rate 1 for simplicity} \]

A, B are fixed, E doesn’t matter

4 reactions \( R_1 = A \quad R_2 = B X \quad R_3 = X^2 Y \quad R_4 = X \)

\[ \dot{x} = R_1 - R_2 - 2R_3 + 3R_3 - R_4 \]

\[ \dot{y} = R_2 - R_3 \]
\[ \frac{dx}{dt} = A - (B + 1)x + x^2y \quad \frac{dy}{dt} = Bx - x^2y \]

So, \( f(x, y) \equiv g(y, x) \)

\[ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \rightarrow \begin{bmatrix} y_2 \\ x_2 \end{bmatrix} \]

\[ \begin{align*}
\frac{d}{dt} & x_2 = y_2 \\
\frac{d}{dt} & x_1 = y_1 \\
\end{align*} \quad \text{since for} \ y
\]

\[ \frac{d}{dt} x_1 = f(x_1, y_1) + \frac{\partial f}{\partial x} (x_2 - x_1) = f_1 \left( \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right) \\
\frac{d}{dt} y_1 = g(x_1, y_1) + \frac{\partial g}{\partial y} (y_2 - y_1) = g_1 \left( \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right)
\]

Note \( f_1(x_1, x_2) = f_2(x_2, x_1) \) so the symmetry holds.

Fixed points \( x_1 = x_2 \quad y_1 = y_2 = v \)

\[ A - (B + 1)u + u^2v = 0 \quad Bu - u^2v = 0 \]

Add together to get \( A - u = 0 \Rightarrow u = A \) +

\[ tv = B/A. \]

With abuse of notation, let we call new \( a, b \) instead.

Matrix \( A = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \quad B = \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} \]

So...

\[ \frac{d}{dt} x = -(b + 1)x + 2xy \quad \frac{d}{dt} = x^2 \\
\frac{\delta x}{\delta y} = b - 2xy \quad \frac{\delta y}{\delta y} = -x^2 \]
\[ A = \begin{bmatrix} b-1 & D_x & a^2 \\ -D_x & 0 & 0 \\ -a^2 & 0 & b \end{bmatrix}, \quad B = \begin{bmatrix} D_x & 0 \\ 0 & D_y \end{bmatrix} \]

\[ A + B = \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix} = M_s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ A - B = \begin{bmatrix} b-1-2D_x & a^2 \\ -b & -a^2-2D_y \end{bmatrix} = M_D \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ \text{Tr } M_s = b-a^2-1, \quad \text{Det } M_s = a^2 > 0 \]

\[ \text{Tr } M_D = b-a^2-1 - 2(D_x + D_y) \]

\[ \text{Det } M_D = a^2 - 2(D_y(b-1) - D_x a^2) + 4 D_x D_y \]

\[ \text{Tr } M_D < 0 \Rightarrow \text{Tr } M_0 < 0 \]

Can \( M_0 \) even be negative?

What does \( M_D \) mean?

Assume \( \text{Tr } M_s < 0 \Rightarrow a^2 > b-1 \)

\[ \text{Necessary condition for Det } M_0 < 0 \Rightarrow \begin{cases} D_y(b-1) - D_x a^2 > 0 \Rightarrow b > 0 \end{cases} \]
\[ \Rightarrow \quad \frac{\partial y}{\partial x} = \frac{a^2}{b-1} > 0 \quad \text{since } a^2 > b-1 \]

\[ \begin{xy}
  0 \ar@{<->}[dr] \ar@{<->}[rr] & & Y \ar@{<->}[dl] \\
  X \ar@{<->}[rr] & & 0
\end{xy} \]

\[ \begin{array}{l}
\text{Suppose } \frac{\partial y}{\partial x} = 0 \quad \text{Then choose } \frac{\partial y}{\partial x} \\
\text{large enough so that}
\end{array} \]

\[ a^2 - 2b(y(b-1)) < 0 \]

\[ \text{Thus for small enough } \frac{\partial y}{\partial x} \text{ can be stabilized} \]

\text{to uniform state.}

More generally for 2-dimensional coupled system with diffusion:

\[ A = \begin{bmatrix} \alpha - b & b \\ s & \delta - s \end{bmatrix} \quad B = \begin{bmatrix} D_x & 0 \\ 0 & D_y \end{bmatrix} \]

\text{Two clusters of inhibitor dynamics:}

\[ \begin{xy}
  0 \ar@{<->}[dr] \ar@{<->}[rr] & & Y \ar@{<->}[dl] \\
  X \ar@{<->}[rr] & & 0
\end{xy} \]

\text{Positive feedback} \quad \text{Act - Inhibitor}

\[ \text{Prove } \begin{bmatrix} - & + \\ - & + \end{bmatrix} \text{ cannot have diffusive pattern formation.} \]

\[ \text{(Note } (- -) \text{ same since diagonal product is all that comes out) } \]
HW: Numerically explore Brune's law for

- Let \( a = 0.6 \), \( b = 1.25 \)
- Find a curve in \((D_x, D_y)\) st. \( \text{det} \ M_0 = 0 \)
- Pick \((D_x, D_y)\) st. \( \text{det} \ M_0 < 0 \) & numerically solve \( DGE \). By Guo Max
- There are equilibrium st. \( x_1 \neq x_2 \)
We have studied the case of a pair of coupled cells.

How much can this be generalized?

Let's say we have $N$ "cells"

![Diagram showing interconnected nodes labeled $x_1$, $x_2$, ..., $x_N$.]

...and coupling between them, that is to say a form of coupling, but not necessarily to any element.

Furthermore, suppose that each cell has the same number of arrows going into it.

Since coupling is the same to everyone and have same # arrows, this means that there is symmetry in the response that if we start with $x_1 = x_2 = \cdots = x_N$, then this is invariant.

The homogeneous solution is a solution.

Let's look at the linearization of this system.
Let $A$ be linearize matrix for self $\\&$ $B$ \ for coupling

In above picture:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

Let $Q = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$

This is called **Adjacency matrix**

(can write big matrix using tensors $A \otimes I + B \otimes Q$

but let just keep it simple)
Let \( \vec{e} \) be eigenvector of \( Q \) and \( \lambda \) be the corresponding eigenvalue.

1. Note that \( \lambda = 2 \) and \( \vec{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \) is.

The symmetric eigenvalue.

2. Note \( A - \lambda I \) is for different type of calculation.

Because of their symmetry, we can reduce the system to \( 5 \times 5 \) system.

Claim: eigenvalues of \( M \) are eigenvalues of \( A + \lambda jB \) for \( j = 1, \ldots, 5 \).

Proof: Assume \( \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \\ \vec{e}_4 \\ \vec{e}_5 \end{bmatrix} \) be eigenvector components of eigenvector \( \vec{e} \) with eigenvalue \( \lambda \) of \( Q \).

\[ Q \vec{e} = \lambda \vec{e} \]

Let \( V = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \\ \vec{e}_4 \\ \vec{e}_5 \end{bmatrix} \) \( \in \mathbb{R}^n \),

\[ MV = \begin{bmatrix} A \vec{e}_1 + B \vec{e}_2 + B \vec{e}_3 + B \vec{e}_4 + B \vec{e}_5 \\ A \vec{e}_2 + B \vec{e}_1 + B \vec{e}_3 + B \vec{e}_4 + B \vec{e}_5 \\ A \vec{e}_3 + B \vec{e}_2 + B \vec{e}_1 + B \vec{e}_4 + B \vec{e}_5 \\ A \vec{e}_4 + B \vec{e}_3 + B \vec{e}_2 + B \vec{e}_1 + B \vec{e}_5 \\ A \vec{e}_5 + B \vec{e}_4 + B \vec{e}_3 + B \vec{e}_2 + B \vec{e}_1 \end{bmatrix} \]

Note \( \vec{e}_2 + \vec{e}_5 = \lambda \vec{e}_1 \)

\( \vec{e}_1 + \vec{e}_5 = \lambda \vec{e}_2 \) since eigenvalue of \( Q \).
\[ \hat{\mathbf{v}} = \begin{bmatrix} \mathbf{A} + \lambda \mathbf{B} & \mathbf{U} \\ \mathbf{U}^T & \mathbf{U} \end{bmatrix} \]

So eigenvalues of \( \hat{\mathbf{m}} \) are found by solving \((\mathbf{A} + \lambda \mathbf{B}) \mathbf{v} = \lambda \mathbf{v} \) eigenvalue problem \( \lambda = 1, \ldots, 5 \)

Nothing special about all the "1's" in \( \mathbf{A} \) could be arbitrary.

**Application:** Pulse coupled oscillators

Consider a firefly. He flashes periodically with frequency \( \omega \). With his flash, he is able to communicate to other fireflies.

As with limit cycle oscillators, can be reset by stimuli.

\[ \text{Active: } \Delta(s) := T - T'(s) \]

Called phase resetting curve.

**Let \( \phi_j(t) \):** The phase of the \( j \)th

**Insect** (defined mod \( T \))
Let $\theta_j(t)$ be the pulse of light emitted by the firefly.

\[ \frac{d\theta_j}{dt} = \omega_j + \sum_{n=1}^{N} g(n) \cdot \theta_n(t) \cdot \Delta(\theta_j) \]

Firefly frequency
Input from Firefly's response function

Simplification: $\omega_j = \omega \neq \omega_j$ (identical)

$\sum_{n=1}^{N} g(n) = g \neq g_j$  Let $T = 2\pi$

Synchrony: $\theta_j(t) = \Theta(t)$

\[ \frac{d\theta_j}{dt} = \omega + g \cdot P(\theta) \cdot \Delta(\theta) \]

Assume $\omega + gP(\theta) \Delta(\theta) > 0$ for all $\theta$

This means $\dot{\theta} > 0$ and period is:

\[ \int_{\omega + gP(\theta)\Delta(\theta)}^{P_0} = \text{period} \]

Is synchrony stable?

Let's look at the linearization around $\theta_j(t) = \Theta(t)$
\[ \ddot{Y}_j = \Delta(\Theta(t_1)) \sum_{k=1}^{N} g_{jk} \frac{\partial f(\Theta(t_1))}{\partial \Theta_k} Y_k(+) + g \Delta(\Theta(t_1)) \frac{\partial f(\Theta(t_1))}{\partial \Theta} Y_j(+) \]

Using the theory above, let \(\lambda_2\) be an eigenvalue of \(\Delta(\Theta(t_1)) \frac{\partial f(\Theta(t_1))}{\partial \Theta} \). Note \(\lambda_1 = g\).

\[ \dot{Y} = u(t) e + \]

\[ \dot{U} = [\Delta(\Theta(t)) \lambda_2 \frac{\partial f(\Theta(t))}{\partial \Theta} + g \Delta(\Theta(t)) \frac{\partial f(\Theta(t))}{\partial \Theta}] U \]

\[ \stackrel{U_0}{=} \alpha(t) U(t) \]

If \[ \int_{t}^{t+T} \dot{X}(s) ds \neq 0 \] then \(U(t)\) is unstable since

\[ U(t+T) = U(t) e \]

Let us do a little calculation.

\[ \dot{\Theta} = \omega + g \frac{\partial f(\Theta)}{\partial \Theta} \Delta(\Theta) \]

\[ \dot{\Theta} = [g \frac{\partial f(\Theta)}{\partial \Theta} \Delta(\Theta) + g \frac{\partial f(\Theta)}{\partial \Theta}] \dot{\Theta} \]

Since \(\dot{\Theta}\) is periodic, we have

\[ g \int_{t}^{t+T} \frac{\partial f(\Theta(t))}{\partial \Theta} \Delta(\Theta(t)) \dot{\Theta}(t) dt = 0, \]

so this means

\[ \int_{t}^{t+T} \frac{\partial f(\Theta(t))}{\partial \Theta} \Delta(\Theta(t)) dt = - \int_{t}^{t+T} \rho(\Theta(t)) \Delta(\Theta(t)) dt \]
Thus
\[ \int_{a(s)}^{b} d\sigma = (g - \lambda_0) \int_0^\rho \rho(t) \Delta'(\phi(t)) \, dt \]

Suppose all entries of $G$ are positive.

Then F.P. $\Rightarrow \text{Re}(g - \lambda_0) > 0$ since

1. $|g| > |\lambda_{el}|$ (and $g > 0$ is real).
2. If $\int_0^\rho \rho(t) \Delta'(\phi(t)) \, dt < 0$ then such is stable. For Firefly

\[ \Delta(t) \sim -k \sin \phi \] so \[ \Delta'(t) = -k \cos \phi \]

if $p(t)$ is centered around 0 and positive.

Then

\[ \int_{-\pi}^{\pi} p(t) \Delta'(t) \, dt < 0 \]

Integral will be negative!