Homework 5: Due April ??

1. We have seen bifurcation of squares and rolls in a simple lattice and in a square periodic domain when there is a zero eigenvalue and we have seen bifurcation of waves and standing oscillations in a one-dimensional domain. So, now, suppose we are in a square periodic domain that has all the usual reflection and translation symmetries in space and translation in time but the nullspace is 8-dimensional and has the following form:

\[ z_1 e^{i(t+x)} + z_2 e^{i(t+y)} + z_3 e^{i(t-x)} + z_4 e^{i(t-y)} + c.c \]

Derive the general bifurcation equations for this system. Here is a little help. Rotation of the square takes \( z_1 \rightarrow z_2, z_2 \rightarrow z_3, z_3 \rightarrow z_4, z_4 \rightarrow z_1 \) so all you need to compute is

\[ z'_1 = f_1(z_1, z_2, z_3, z_4) \]

and the rest will follow. Reflection across the x-axis leaves \( z_1 \) invariant but interchanges \( z_2, z_4 \), so this further restricts the number of different coefficients.

Determine algebraic conditions for the following types of solutions

(a) Traveling rolls \((z_1 \neq 0, z_2 = z_3 = z_4 = 0)\) and determine when they are stable

(b) Standing rolls \((z_1 = z_3 \neq 0, z_2 = z_4 = 0)\)

(c) What does the pattern \(z_1 = z_2 \neq 0, z_3 = z_4 = 0\) look like? and is it a solution?

(d) What does \(z_1 = z_2 = z_3 = z_4\) correspond to and is it a solution?

(e) Alternating rolls - \(z_1 = z_3 = -iz_2 = -iz_4\). Sketch these - they are really cool!

2. Consider the phase equation:

\[ \theta_t = a(x) + \theta_x^2 + \theta_{xx} \]

with boundary conditions

\[ \theta_x(0, t) = \theta_x(1, t) = 0. \]

This describes the evolution of the phase in the presence of spatial heterogeneities, \(a(x)\). Assume \(a(x)\) is continuous in \([0, 1]\). Let \(\theta(x, t) = \omega t + \int u(y), dy\) so that

\[ \omega = a(x) + u^2 + u_x. \]

Prove that there is some value of \(\omega\) such that this equation has a solution satisfying \(u(0) = u(1) = 0\). (There are several approaches to this; one is to make the Cole-Hopf transformation, \(u = v_x/v\) and convert it to a Sturm-Liouville eigenvalue problem. But the most direct way is to use shooting.) Solve the BVP numerically for \(a(x) = \exp(-4(x - 1/2)^2)\) and estimate \(\omega\).
3. In class, we derived Burgers equation for the evolution of phase:

\[ \theta_t = \alpha \theta_x^2 + \beta \theta_{xx} \]

where

\[ \alpha = <U^*(t)DU''(t)>, \quad \beta = <U^*(t)DU'(t)> \]

with \( U, U^* \) the oscillation and its adjoint and \( D \) the diffusion matrix.
Consider the CGL in rectangular coordinates:

\[
\begin{align*}
    u_t &= u(1 - u^2 - v^2) - vq(u^2 + v^2) + u_{xx} - dv_{xx} \\
    v_t &= v(1 - u^2 - v^2) + uq(u^2 + v^2) + v_{xx} + du_{xx}
\end{align*}
\]

Given \( U(t) = (\cos qt, \sin qt) \) and \( U^*(t) = (\cos qt - (1/q) \sin qt, \sin qt + (1/q) \cos qt) \), compute \( \alpha \) and \( \beta \). Under what circumstances is diffusion, \( \beta \) positive?

4. Discretize the above equation with \( dx = 1 \) into 100 bins (total domain length is 100) with periodic boundary conditions. (In case you forget,

\[
    u_{xx} \approx \frac{u[j-1] - 2u[j] + u[j+1]}{dx^2}
\]

and periodic boundary conditions mean \( u[0] = u[100] \) and \( u[101] = u[1] \). Solve the equations with random initial conditions and choose \( d = 1 \). Try \( q = 2 \) and \( q = 0.5 \). XPP code is included on the web page. You should integrate for a long period of time and see the result of the diffusive instability.