More math stuff: Public goods games

Public goods games are games where there are a bunch of cooperators + some defectors. If N guys interact, then each cooperator increases the common resource by rc at a cost c. The total is divided by all N, so if there are K cooperators, then $P_D(h) = \frac{rc}{K}$, $P_C(h) = P_D(h) - c$, so always better to defect as usual.

If $rc > 1$ then $P_C(N) < 0 = P_D(0)$ so cooperation is doomed. In any interaction group cooperators always fare worse, so on a population wide scale need to look at different sized groups.

We will now explore the following scenario based on placing these on an ecological setting. Let $N$ be the population of cooperators, $V$, defectors, where we can have born go extinct leaving empty space $w$. $(w = 1 - u - v)$

$$\frac{du}{dt} = u \left[w(f_c + b) - d\right], \quad \frac{dv}{dt} = v \left[w(f_d + b) - d\right]$$

available space death fitness birth

If $f_c = f_d = 0$, $u_t = u(\theta(1 - u - v) - d)$, $v_t = v(\theta(1 - u - v) - d)$ and $u + v = 1$ is equil.

So, what are $f_c, f_d$?

$u$ = prob of finding a coo; $v$ = prob of defector; $w$ = failure to find a partner ($u + v$) = average group size. An individual finds itself in a group of size $S$ with probability $\binom{N}{S} \left(\frac{u}{u+v}\right)^m \left(\frac{v}{u+v}\right)^{S-1}$. Since $w = u + v + S - 1$, $u + v = 1$.

In my group you face $m$ coops + $S-1-m$ defectors with prob:

$$\binom{S-1}{m} \left(\frac{u}{u+v}\right)^m \left(\frac{v}{u+v}\right)^{S-1-m}$$

Payoff for detecting is $\frac{r}{S} \sum_{m=0}^{S-1} \binom{S-1}{m} m \left(\frac{u}{u+v}\right)^m \left(\frac{v}{u+v}\right)^{S-1-m} = P_D(S)$

$P_C(S) = P_D(S) + \frac{c}{S-1}$, where assume $c = 1$ WLOG.
We average this over all sizes \( n \rightarrow N \):
\[
\bar{f}_j = \sum_{s=2}^{N} \frac{(N-1)}{(s-1)} (1-w)^{s-1} w^{N-s} p_i(s)
\]
\[
f_D = r \frac{w}{1-w} \left( 1 - \frac{1-w^N}{N(1-w)} \right) \quad f_C = f_D - \bar{F}(w)
\]
\[
F(w) = 1 + \frac{(r-1) w^{N-1}}{N} - \frac{r}{N} \left( \frac{1-w^N}{1-w} \right)
\]

So this is pretty cool. First note that if there are no cooperators then \( f_B = 0 + \bar{v} = V(WB - d) \). Now if we assumed \( d > b \), then detectors will die!!

So we assume \( d > b \), thus the only way detectors will survive is if there are cooperators around.

With this assumption \( (u = v = 0) \) is always A.S.

Of course there is also a state with no detectos, \( V = 0 \) so it turns out a state of coexistence. We fix \( N = 8 \), \( b = 1 \), \( d = 1.2 \) and vary \( r \), thus is the reward for cooperation.

As \( r \) decreases from a TIB but is subcritical and unstable. This is an example of an activator/inhibitor system.
\[ \begin{bmatrix} \alpha & -\beta \\ \gamma & -\delta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha u - \beta v \\ \gamma u - \delta v \end{bmatrix} \]

Slope of \( v \)-nullcline \( (v = \frac{\partial H}{\partial u}) \) is \( \frac{\delta}{\beta} \)

Slope of \( u \)-nullcline \( (v = \frac{\partial H}{\partial v}) \) is \( \frac{\alpha}{\beta} \)

From picture, \( \frac{\delta}{\beta} > \frac{\alpha}{\beta} \) \( \Rightarrow \gamma \beta > \alpha \delta \) \( \Rightarrow \text{det} > 0 \)

Note \( + \overrightarrow{u} \Rightarrow \text{Activator/Inhibitor} \)

\text{SPATIAL MODEL}

\[
\frac{\partial u}{\partial t} = f(u,v) + D_u \frac{\partial^2 u}{\partial x^2} \quad \text{Periodic BC, eg or} \quad \text{NO Flux} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0
\]

\[
\frac{\partial v}{\partial t} = g(u,v) + D_v \frac{\partial^2 v}{\partial x^2} \quad \text{Periodic BC, assume} \quad \text{NO Flux}
\]

\[
\text{GUESS} \quad u(x,t) = U^* e^{\lambda t} \cos \frac{\pi n x}{L} \quad v(x,t) = V^* e^{\lambda t} \cos \frac{\pi n x}{L}
\]

Note \( u(x,0) = u(L,t) = 0 \) as required. Define \( k = \frac{\pi n}{L} \) \( u_{xx} = -k^2 u, v_{xx} = -k^2 v \) so

\[
\begin{align*}
\lambda u^* &= \alpha u^* - \beta v^* - D_u k^2 u^* \\
\lambda v^* &= \gamma u^* - \delta v^* - D_v k^2 v^*
\end{align*}
\]
\[ \lambda (w^*, v^*) = \begin{bmatrix} \alpha - Dv k^2 & -\beta \\ -\gamma & -\delta - Dv k^2 \end{bmatrix} (w^*, v^*) \]

Re \( \lambda < 0 \) iff det > 0, \( \text{Tr} < 0 \)

\( \text{Tr} = \alpha - \gamma - (D_n + D_v) k^2 \)

Since \((\bar{w}, \bar{v})\) is stable solution of the space-independent model (\(k = 0\)), we have \(\alpha - \gamma < 0\) and \(\beta \gamma > \alpha \delta \)

\( \Rightarrow \text{Tr}(k) < 0 \) if \( \text{Tr} > 0 \) since \(D_n, D_v\) are positive

\( \text{det} = D_n D_v k^4 - (\alpha D_v - \delta D_u) k^2 + \beta \gamma - \alpha \delta \)

For \(k \to O\) or large, \(\text{det}(k) \to 0\) \(\Rightarrow\) stable

but what about intermediate values of \(k\)?

If \(\alpha D_v - \delta D_u\) is large and positive, then

we can get \(k^2\) st \(\text{det} < 0\) so those modes will grow.

\(\alpha D_v - \delta D_u >> 0 \Rightarrow \frac{D_v}{D_u} >> \frac{\delta}{\alpha} \geq 1\)

So this says to get spatial growth (larry form)

need \(v\) to wander more than \(u\)

"operators stick together" detectors wander around to find more sucked to exploit!"
\[ D_V = 10 \, D_u \]
\[ b = 1, \, d = 1.2, \, N = 8, \, r = 2.5 \]

- **Front**
  - \( r = 2.3 \)
  - \( D_V = 0.25 \)
  - \( D_u = 0.75 \)
  - Bistable

- **pulse**
  - \( r = 2.3 \)
  - \( D_V = 0.25 \)
  - \( D_u = 125 \)

- **dead**
  - \( r = 3 \)
  - \( D_V = 0.25 \)
  - \( D_u = 0.75 \)
  - Monostable

- **hanging around**
  - \( r = 2.45 \)
  - \( D_V = 0.25 \)
  - \( D_u = 0.75 \)

- **chaos**
  - \( b = 0.7, \, d = 0.3 \)
  - \( r = 2.45 \)
  - \( D_V = 0.25 \)
  - \( D_u = 0.75 \)
Return to simple replicator plus space:

Hubon + Vickers. Need to be a little careful since with diffusion, cannot be sure that at each spatial location, the total is conserved. Thus we replace the usual replicator dynamics by:

\[ u_i \left( \frac{(Au)_i - U^TAU}{N} \right) \]

where \( N = \sum u_i \). In absence of space, \( N = 1 \) is conserved.

In keeping with Vickers paper, \( A = \begin{bmatrix} \alpha & 0 \\ \beta & 0 \end{bmatrix} \)

so in absence of space. We let \( u_1 = u \), \( u_2 = v \)

\( f_u = \alpha u \), \( f_v = \beta v \)

\( \rho = \alpha u^2 + \beta v^2 \)

so get:

\[ u \left( \frac{\alpha u}{u+v} - \frac{\alpha u^2 + \beta v^2}{(u+v)^2} \right) = u \frac{u\alpha u^2 + \alpha uv - \alpha u^2 - \beta v^2}{(u+v)^2} \]

\[ = \frac{uv}{(u+v)^2} (\alpha u - \beta v) \]

for \( v \), \( \frac{v}{(u+v)^2} (\alpha u - \beta v) \)

\[ \frac{\partial u}{\partial t} = \frac{uv}{(u+v)^2} (\alpha u - \beta v) + D_u u_{xx} \]

\[ \frac{\partial v}{\partial t} = -\frac{uv}{(u+v)^2} (\alpha u - \beta v) + D_v v_{xx} \]

Suppose \( D_u = D_v \). Then \( u + v \rightarrow \text{constant}, \text{say} \]

1
\[ \frac{\partial u}{\partial t} = u(1-u) \left[ \alpha u - \beta (1-u) \right] + D \frac{\partial^2 u}{\partial x^2} \]
\[ = u(1-u) \left[ (\alpha + \beta) u - \beta \right] = (\alpha + \beta) u(1-u)(u-c), \quad c = \frac{\beta}{\alpha + \beta} \]

By rescaling time and space, we can get rid of \( \alpha + \beta \).

\[ u_t = u_{xx} + u(1-u)(u-c) \]

Bestable (assume \( \log \( c < \frac{1}{2} \) \))

Look for traveling wave:

\[ u(x,t) = U(x - \theta t) \]
\[-\theta U' = U'' + f(u) \]
\[ U' = W, \quad W' = -\theta W - f(u) \]

\( \theta \) small and positive

\( \theta \) large and positive

\( \theta \) just right
Extortion strategy or Zero Determinant (ZD) strategy.

Here, we return to the iterated PD models with \( (R, S, T, P) \) as usual, e.g. \( (3, 0) \).

Press and Dyson show first, an interesting result: If a short memory player
plays a long memory player, The short memory players score is the same as if
the long memory guy played a short memory strategy. Because of this
result, we can derive strategies for \( X \) (short mem) assuming each player has
memory 1!

Let \( X, Y \) be random variables with values \( x, y \) that are the players respective
memory on a given iteration. Suppose player \( X \) keeps history \( H_0 \) but \( Y \) keeps
longer history \( H_1 \).

We want to show that the joint probability distribution of \( (X, Y) \) given history
\( (H_0, H_1) \) averaged over \( (H_0, H_1) \) is same as joint prob of \( (X, Y) \) averaged
over shorter history \( H_0 \)

\[
\langle P(X, Y | H_0, H_1) \rangle \text{ is } \sum_{H_0, H_1} P(X, Y | H_0, H_1) P(H_0, H_1)
\]

\[
= \sum_{H_0, H_1} P(x | H_0) P(y | H_0, H_1) P(H_0, H_1) \text{ [Definition of conditional]} \]

\[
= \sum_{H_0} P(x | H_0) \left[ \sum_{H_1} P(y | H_0, H_1) P(H_1 | H_0) P(H_0) \right] \text{ [sum over } H_1 \text{]} \]

\[
= \sum_{H_0} P(x | H_0) \left[ \sum_{H_1} P(y | H_1, H_0) P(H_1 | H_0) P(H_0) \right] \text{ [sum over } H_1 \text{]} \]

\[
= \sum_{H_0} P(x | H_0) P(y | H_0) P(H_0) \text{ [Bayes]} \]

\[
= \langle P(X | H_0) \rangle_{H_0}
\]

Intuitively- from \( X \)’s point of view, \( X \) views \( Y \)’s long strategy as a peculiar
random number generator. Thus \( X \) player with the shortest memory sets the rules
of the game.
ZD Strategies:
As usual let $x, y \in \{c, d, d', d''\}$ where $c, d$ are cooperate/defect
$X$'s strategy is $\pi = (p_1, p_2, p_3, p_4)$ are probability to cooperate given the outcome $\omega = (q_1, q_2, q_3, q_4)$ are $Y$'s strategy, seen from his perspective, i.e. $y = \{c, d, d', d''\}$ (Note this is a little different than we used before. So $p = 1 - $)

\[
\begin{bmatrix}
(p_1, p_1, (1-\pi), (1-\pi)) \\
(1-\pi), p_2, p_3, (1-\pi)) \\
(1-\pi, 1-\pi, p_4, (1-\pi))
\end{bmatrix}
\]

As before we get a Markov transition matrix:

\[
M = \begin{bmatrix}
p_1 & p_1 & (1-\pi) & (1-\pi) \\
p_2 & (1-\pi) & p_3 & (1-\pi) \\
p_3 & (1-\pi) & p_4 & (1-\pi) \\
p_4 & (1-\pi) & (1-\pi) & (1-\pi)
\end{bmatrix}
\]

$V$ should be an $n-1$ left eigenvector: $V^T M = V^T \Rightarrow V^T M^\prime = 0$

$M^\prime = M - I \Rightarrow \det M^\prime = 0$

Recall Cramer's Rule: $\text{Adj}(M^\prime) M^\prime = \det(M^\prime) I = 0$

$\text{Adj}(M^\prime) = \text{matrix of } M^{11}, (3x3)$ determinants, since $V^T M^\prime = 0$

This means every row of $\text{Adj}(M^\prime)$ is proportional to $V$

\[
\begin{bmatrix}
h_{11} & h_{12} & h_{13} & h_{14} \\
h_{21} & h_{22} & h_{23} & h_{24} \\
h_{31} & h_{32} & h_{33} & h_{34} \\
h_{41} & h_{42} & h_{43} & h_{44}
\end{bmatrix}
\]

so, for example, up to sign, the components of $V$ are the fourth row of $\text{Adj}(M^\prime)$, which are the cofactors of the fourth column of $M^\prime$ which are just the determinants of the first three columns of $M^\prime$ leaving out the relevant row. These determinants are unchanged by adding the first column of $M^\prime$ to the second and third columns.
Let $\vec{f}$ be arbitrary v.c. for $\vec{r}$. Then $\vec{V} \cdot \vec{f}$ is easy to compute using the above manipulation:

$$\vec{V} \cdot \vec{f} = \det \begin{bmatrix}
-1 + p_4 & -1 + p_1 & -1 + q_1 & f_1 \\
-1 + p_2 & q_2 & q_3 & f_2 \\
q_3 & q_2 & -1 + q_4 & f_3 \\
q_4 & q_3 & q_2 & -1 + q_1 \\
\end{bmatrix} = D(p, q, f)$$

$$\vec{V} \cdot \vec{f} = \frac{\vec{p} \cdot \vec{V}}{\vec{q} \cdot \vec{V}}$$

Note that two columns depend only on one player's strategy!

$$\vec{s}_x = \begin{bmatrix} p_1 \\ s_1 \\ q_1 \end{bmatrix}, \quad \vec{s}_y = \begin{bmatrix} r_1 \\ s_2 \\ q_2 \end{bmatrix}$$

$$s_x = \frac{\vec{V} \cdot \vec{s}_x}{\vec{V} \cdot \vec{V}}, \quad s_y = \frac{\vec{V} \cdot \vec{s}_y}{\vec{V} \cdot \vec{V}}$$

$$\alpha \vec{s}_x + \beta \vec{s}_y = \frac{D(p, q, \alpha \vec{s}_x + \beta \vec{s}_y + \vec{V})}{D(p, q, \vec{V})}$$

If I choose $\vec{p} = R \begin{bmatrix} \alpha \vec{s}_x + \beta \vec{s}_y + \vec{V} \end{bmatrix}$, then $D(p, q, \vec{V})$ vanishes since two columns are proportional and this means I enforce a linear relationship between the payoffs no matter what $q$ is!!

Of course, $\alpha \vec{s}_x + \beta \vec{s}_y$ may not be feasible since $p_i \in [0, 1]$

**Example:** X sets Y's score!

Set $\alpha = 0$. Use $\vec{p} = \beta \vec{s}_y + \vec{V}$ to solve for $p_2, p_3$ in terms of $p_1, p_y$.

Elimination:

$$p_2 = \frac{p_1 (T-p) - (1+p_y) (T-R)}{R-P}, \quad p_3 = \frac{(1-p_1) (p-s) + p_y (R-s)}{R-P}$$

To force $s_y = \frac{(1-p_1) p + p_y R}{(-p_1) + p_y}$ in $0 < T > R > p > s$.

So there is a feasible strategy when $p_1 \leq 1$, $p_y > 0 \Rightarrow p_2 \leq 1$, $p_3 \leq 0$. Clearly $s_y$ is weighted average of $1-p_1 + p_y$ so all possible scores between $s + R$ are possible for Y. X can completely ignore Y but set Y's score. X can spoof Y's then play a better strategy.
\[ p = \alpha S_x + \gamma \frac{r}{p} \]
\[ p_2 = \frac{(1+\phi)(p-S)+\phi(p-S)}{p-S} \geq 1 \quad p_3 = -\frac{(1-\gamma)(T-p)-\gamma(T-p)}{p-S} \leq 0 \]

Only one feasible point: \((1,1,0)\) which is always cooperate. So \(X\) cannot unilaterally set his score.

"Extortion" Suppose \(p = \phi \left[ (S_x - p) - \chi (S_y - p) \right] \). Then this means \(\phi \left[ (S_x - p) - \chi (S_y - p) \right] = 0 \iff (S_x - p) = \chi (S_y - p) \iff X\) can make his gain above mutual determ (p) \(X\) times greater than \(Y\)'s.

Solving the equation for \(\phi_i\):

\[ \phi_i = 1 - \phi \left( 1 - \frac{p-S}{p-S} \right) \quad \phi_2 = 1 - \phi \left( 1 + \chi \frac{T-p}{p-S} \right) \quad \phi_3 = \phi \left( 1 - \chi \frac{T-p}{p-S} \right) \]

so possible strategies exist for any \(X\) + small \(\phi\), e.g.

\[ 0 \leq \phi \leq \frac{(p-S)}{(p-S) + \chi (T-p)} \]

Clearly \(X\)'s score depends on \(Y\)'s strategy + both are maximized when \(Y\) cooperates: \((1,1,1,1)\) in which case:

\[ S_x = \frac{p(T-R) + T(R-S) - p(T-p)}{(T-R) + \chi (R-S)} \quad (X=1 \Rightarrow S_x = S_y = R) \]

E.g. \((5,3,1,0)\)

\[ \tilde{p} = \left[ 1 - 2 \phi (X_1), 1 + \phi (X_4), 0 \right] \quad \text{ok for } 0 \leq \phi \leq (X_4)^{-1} \]

\[ S_x = \frac{2+3X}{2+3} \quad S_y = \frac{12+3X}{2+3X} \quad \text{Mutual 100% \iff 3} \quad S_x > 3 + \chi \quad S_y < 3 \]

As \(X \to 0\), \(S_x \to \frac{10}{3} + S_y \to \infty\) so \(Y\) has no reason to cooperate so \(X\) should not get too greedy!

\[ S_{ay} X = 3 + \frac{1}{3} \text{ midpoint of feas. br.} \quad \tilde{p} = \left( \frac{11}{13}, \frac{5}{13}, \frac{7}{26}, 0 \right) \]

\[ S_x \approx 3.73, \quad S_y \approx 1.91 \quad \text{Note } X = 1, \phi = 1/9 \Rightarrow TFT \]

\((1,0,1,0)\)
Spatial games in a continuum

This is my version of a spatial game. We will later look at some others. Let consider a two strategy game with a 2x2 pay off matrix \( \begin{pmatrix} R & S \\ T & P \end{pmatrix} \) (not necessarily PD).

Let \( u(x,t) \) be density of players using strategy C, so that \( v(x,t) = 1 - u(x,t) \) is density of players using strategy D. Now here is how I do the spatial game. Let me look at a weighted neighborhood around \( x \) and compute the fitness if I played C:

\[
 f_c(x) = R \int w(y) u(x-y,t) dy + S \int w(y) v(x-y,t) dy
\]

and fitness if I played D:

\[
 f_d(x) = T \int w(y) u(x-y,t) dy + P \int w(y) v(x-y,t) dy
\]

Here \( w(y) \) is a symmetric weighting function, say exponential or Gaussian.

I will go from C \( \rightarrow \) D at a rate \( \beta \) and from D \( \rightarrow \) C at a rate \( \alpha \) where

\[
 \alpha = H(f_c - f_d), \quad \beta = H(f_d - f_c), \quad \text{where} \ H \ \text{is some monotonically non-negative function, e.g.} \ 1/(1+e^{-x})
\]

so, e.g. if \( f_c > f_d \), then \( \alpha \) will be bigger than \( \beta \) and there will be a loss of D and gain of C at x:

\[
 \frac{\partial u}{\partial t} = \alpha (v - u) - \beta \left( u(1-u) - u \right)
\]

\[
 \alpha = H(f_c(x) - f_d(x)), \quad \beta = H(f_d(x) - f_c(x))
\]
It is now clear how to do this for m strategy games
Let $A = (a_{ij})$ be payoff matrix. Let $u_i(x,t)$ be density playing strategy $i$. Then

$$f_i(x,t) = \int w(y) \sum a_{ij} u_j(x-y,t) \, dy$$

$$\alpha_{ij} = \frac{H(f_i - f_j)}{r_i - r_j}$$

$$\frac{du_i}{dt} = \sum \alpha_{ij} u_j \quad \alpha_{ij} = H(f_i(x,t) - f_j(x,t))$$

This is not like replicator dynamics, which we will look at later.
It is a new model. Let's look at equilibria for $n=2$

$$\frac{du}{dt} = a(1-u) - bu = 0 \Rightarrow u = \frac{a}{a+b}$$

$$\alpha = H(a_{11} u + a_{22} (1-u) - a_{21} u - a_{12} (1-u))$$

$$= H[(a_{11} - a_{21} + (a_{22} - a_{12}) u)]$$

$$= H(c - (d-c)u)$$

$\beta = H(-c + (d-c)u)$ so as expected all that matters is the off-diagonal terms. So when we can assume $A = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

For $d/c > 0$, $c < 0$. For bistable game: $c,d < 0$

so let's try $c = -1, d = -4$ for example $r = 10$

$$u$$

3 roots bistable

Suppose $w(x)$ is exponential $+$. There are just two strategies

$u$ in our bistable game

$$f_1 = -c w(x) (1-u), \quad f_2 = -d w(x) u$$

$$= -c + c w(x) u$$
Let $z(x,t) = \int_{-\infty}^{\infty} w(x-y) u(y,t) \, dy + \text{boundary terms}$ with $u(x) = \frac{1}{2} e^{-|x|}$.

Then it is easy to show that:

$z(x,t) - \frac{\partial^2 z}{\partial x^2} = u(x,t)$

so we have

$f_1 - f_2 = -C + C w \frac{\partial u}{\partial x} w \frac{\partial u}{\partial x} u = -C + (d+c) z$

$f_2 - f_1 = C - (c+d) z$

$\frac{\partial u}{\partial t} = H(-c + (c+d) z) (1-u) - H(c - (c+d) z) u$

Fixed point and $\partial_t x, t$ are $z = u$

which has 3 equal $u_0 < u_1 < u_2$

Numerical solution

![Graph showing space and time with u0 and u2 values](image)

Looks like travelling wave joining $u_0$ to $u_2$

Let $u(x,t) = \Phi(x - \theta t)$, $\frac{\partial }{\partial t} = \Phi(x - \theta t)$

$-\theta \frac{\partial u}{\partial z} = H(-C + (d+c) z) (1-u) - H(C - (c+d) z) u$

$\frac{\partial^2 z}{\partial x^2} = z - u$

$20 \times \Delta x \approx 0.86$ is velocity

$\frac{\Delta t}{4.5}$
Look for solution $u_2 \rightarrow \theta \rightarrow u_0$.

This is a heteroclinic orbit in many cases of D.S.

$u' = -\frac{E(w, z)}{\theta}$, $z' = w$, $w' = z - u$

$u(0) = (u_0, u_0, 0) \rightarrow (u_1, u_1, 0)$

$u(\pm \infty) = (u_2, u_2, 0)$, $\mathcal{U}(\pm \infty) = (0, 0, 0)$

Linearize about $(u_0, u_0, 0)$, find $2 + 1$ real eigenvalues.

And around $(u_2, u_2, 0)$, $2 + 1$ real eigenvalues.

1-stable, 2D unstable.

1D stable manifold.