 Weakly coupled oscillators

1 Basic theory

Consider the following system:
\[
\frac{dX_j}{dt} = F(X_j) + \epsilon G_j(X_j, X_k)
\]  
(1)
and assume that \(X' = F(X)\) has an asymptotically stable limit cycle solution, \(U(t)\) with period \(T\). Let \(A(t) = D_X F(U(t))\) be the periodic matrix associated with the linearization of \(F\) about the limit cycle. The system
\[
\frac{dY}{dt} = A(t)Y
\]
has a unique periodic solution (up to scalar multiplication), \(Y(t) = U'(t)\). Almost all initial data \(Y(0) = Y_0\) converge to \(U'(t)\) (that is, all data such that \(Y_0^T U'(0) \neq 0\)). Associated with the linear operator, \(LY = dY/dt - A(t)Y\) is the adjoint, \(L^*\). Here we work in the space of (square-integrable) \(T\)-periodic functions with the inner product
\[
(U, V) = \int_0^T U(t)V(t) \, dt.
\]
Under this inner product, the adjoint (recall that if \(L\) is a linear operator, then the adjoint, \(L^*\) is such that \((LU, V) = (U, L^*V)\) for all \(U, V\) is just
\[
L^*Y = -dY/dt - A^T(t)Y.
\]
Since \(Lu'(t) = 0\), this means that \(L\) has a one-dimensional nullspace. Thus, \(L^*\) also has a one-dimensional nullspace, spanned by \(W(t)\) and normalized so that \(W^T(t) U'(t) = 1\). This function, \(W(t)\), is crucial to all our calculations and is called the adjoint. Finally, recall the Fredholm alternative which says that \(Lu = b\) has a solution if and only if \((u^*, b) = 0\) for all solutions to \(L^*u^* = 0\).

Now, suppose that \(X_j(t) = U(t + \theta_j(\tau)) + \epsilon V_j(t) + \ldots\) where \(\tau = \epsilon t\) is a slow time scale and \(U\) is the \(T\)-periodic solution to \(U' = F(U)\). Note that from the chain rule:
\[
\frac{dX_j}{dt} = U'(t + \theta_j)(1 + \epsilon \frac{d\theta_j}{d\tau}) + \epsilon \frac{dV_j}{dt} + \ldots
\]
We plug this series approximation into equation (1) and get to order \(\epsilon\)
\[
\frac{dV_j}{dt} - A(t + \theta_j)V_j = -U'(t + \theta_j)\frac{d\theta_j}{d\tau} + G_j(U(t + \theta_j), U(t + \theta_k)).
\]
There is a \( T \)-periodic solution \( V_j(t) \) if and only if the right-hand side is orthogonal to the adjoint, \( W(t + \theta_j) \), thus, we must have:

\[
0 = \int_0^T W^T(t + \theta_j) \left[ -U'(t + \theta_j) \frac{d\theta_j}{d\tau} + G_j(U(t + \theta_j), U(t + \theta_k)) \right] \, dt.
\]

Recall that \( W^T(t)U'(t) = 1 \), so that we obtain:

\[
\frac{d\theta_j}{d\tau} = \frac{1}{T} \int_0^T W^T(t + \theta_j) G_j(U(t + \theta_j), U(t + \theta_k)) \, dt.
\]

Using periodicity of all the actors involved, this simplifies to

\[
\frac{d\theta_j}{d\tau} = H_j(\theta_k - \theta_j)
\]

where

\[
H_j(\psi) = \frac{1}{T} \int_0^T W^T(t) G_j(U(t), U(t + \psi)) \, dt
\]

(2)

This is the “fundamental equation for weak coupling.” If two oscillators are coupled identically to each other and only differ in their intrinsic properties, we often write

\[
H_j(\psi) = \omega_j + H(\psi)
\]

separating out the heterogeneity from the coupling. So, for a pair of oscillators, we have

\[
\frac{d\theta_1}{d\tau} = \omega_1 + H(\theta_2 - \theta_1), \quad \frac{d\theta_2}{d\tau} = \omega_2 + H(\theta_1 - \theta_2).
\]

Subtracting them and letting \( \phi = \theta_1 - \theta_2 \), we get

\[
\frac{d\phi}{d\tau} = \delta + H(-\phi) - H(\phi) \equiv \delta - 2g(\phi)
\]

where \( \delta = \omega_1 - \omega_2 \) and \( g(\phi) \) is the odd part of \( H(\phi) \). Fixed points of this differential equation correspond to periodic solutions of the original equation and if they are stable fixed points, then the corresponding periodic orbits are also stable. Suppose that \( \delta = 0 \), then, we need only look at the zeros of the odd function \( g(\phi) \). All continuous odd periodic functions have at least two zeros, \( \phi = 0 \) called synchrony and \( \phi = T/2 \) called antiphase. Synchrony is stable if \( g'(0) > 0 \) which means that \( H'(0) > 0 \).

What is the physical interpretation of the adjoint, \( W(t) \)? To see this consider the following:

\[
\frac{dX}{dt} = F(X) + ccE_\alpha \delta(t - s)
\]

where \( E_\alpha \) is the \( \alpha^{th} \) column of the identity matrix, and \( c \) is the physical dimension of the stimulus That is, \( E_\alpha \) is one the standard basis vectors in \( R^n \).

Writing \( X(t) = U(t + \theta) \), we find that

\[
\frac{d\theta}{d\tau} = c\delta(t - s)W_\alpha(t)
\]
Integrating across the delta function, we see that

$$\theta(s^+) = \theta(s^-) + cW_\alpha(s).$$

That is, the phase of the oscillator is perturbed by an amount $cW_\alpha(s)$ when there is an instantaneous perturbation of the $\alpha^{th}$ component of the oscillator. Experimentalists use brief pulsatile stimuli to probe the phase-resetting characteristics of oscillators. Thus, they are able to experimentally compute certain components of the adjoint of the oscillation without even knowing the dynamics! For example, suppose that one adds enough current to a neuron to cause it to repetitively fire with period $T$. Then briefly inject a small pulse of current at a time $t$ after the last spike and measure the time of the next spike, say $T'(t)$. Define $\Delta(t) = T - T'(t)$ to be the time shift called the Phase resetting curve or PRC. This (up to a scalar factor arising from having to divide by $C_m$ the capacitance; recall that the equations for the voltage are $C_m dV/dt = I$) is an approximation of the voltage-component of the adjoint since current injection only directly perturbs the voltage. What are the units of the adjoint in this case? Note that $W^T(t)U'(t) = 1$, so the voltage component of the adjoint has dimensions of time per volt. What is the dimension of $c$? It is current divided by capacitance which is volts per time. Thus the change in timing (or phase) which has dimensions of time (since $\theta_j$ is added to $t$) is $\Delta(t) = TcW_V(t)$ where $W_V(t)$ is the voltage component of the adjoint.

This means, that to use weakly coupled oscillator theory, all we would need to know is the form in which one neuron interacts with another. Consider a
general neural model which receives a weak synapse from another neuron:

\[ C_m \frac{dV_j}{dt} = -I_{ion,j}(V_j, \ldots) - g_{syn}s_k(t)(V_j - V_{syn}) \]

Then the interaction function is just

\[ H(\phi) = \frac{g}{C_m} \frac{1}{T} \int_0^T W_V(t)(V_0(t) - V_{syn})s(t + \phi) \, dt \]

(Note the dimensions: \( g/C_m \) is time\(^{-1} \) and \( W_V(V_0 - V_{syn}) \) has dimensions of time, which cancels, so the whole thing is dimensionless as required since the left hand side is \( d/d\tau \) which is dimensionless.) Note that the parameter \( \omega_j \) is also dimensionless. Suppose this represents a small current, \( I \), added to the neuron. Then

\[ \omega = \frac{I}{C} \frac{1}{T} \int_0^T W_V(t) \, dt \]

and since \( I/C \) is time per volt and \( W_V \) is volt per time, the whole thing is also dimensionless as required. So, for a real neuron, if we know the voltage trace, \( V(t) \) and we have computed the PRC, and we have a model of the conductance of the synapse, \( g_s(t) \), then we can compute an interaction function from these experimental data. The PRC for a cortical cell is shown above.

2 Various adjoints.

It is clear from the above theory that the key to computing the interaction function \( H \) is to compute the adjoint of the periodic orbit. There are a limited number of oscillatory systems where one can actually compute a formula for the adjoint. I describe them below.

2.1 Circle models.

Consider

\[ x' = f(x) \]

where \( f(x) > 0 \). Then in the homework, I ask you to prove that the adjoint is \( 1/u'(t) \) where \( u(t) \) is the solution to the scalar problem with \( u(0) = 0 \) and \( u(T) = 1 \). Let’s apply this to the leaky integrate and fire model (LIF):

\[ \frac{dx}{dt} = (I - x)/\tau \]

The solution to this is \( u(t) = I(1 - \exp(-t/\tau)) \) so that the adjoint is just \( u^*(t) = \tau \exp(t/\tau)/I \). Thus, for a synaptically coupled LIF, with synapse \( s(t) \), we have

\[ H(\psi) = \frac{1}{T} \int_0^T u^*(t)s(t + \psi) \, dt. \]
It is interesting to note that for any system near a saddle-node bifurcation, the oscillator is in some sense equivalent to the quadratic integrate and fire (QIF) system. Here is a coupled pair:

\[ x'_j = a^2 + x_j^2 + \epsilon s_k(t) \]

with \( x_j(0) = -\infty \) and \( x_j(T) = +\infty \) when \( \epsilon = 0 \). Here \( s(t) \) is the effect of one cell firing on the other. In the homework, I ask you to compute the adjoint for this. It turns out to be proportional to \( 1 - \cos(2\pi t/T) \). Thus, for a QIF model,

\[ H(\psi) = K \int_0^T (1 - \cos(2\pi t/T))s(t + \psi) \, dt \]

where \( K > 0 \) is a positive constant. We can simplify this to

\[ H(\psi) = A + B \cos 2\pi \psi / T + C \sin 2\pi \psi / T. \tag{3} \]

Hence, for identical cells, synchrony is stable if and only if \( C > 0 \).

Here is another way to approach the circle model that is sometimes instructive. Consider:

\[ x' = f(x) + g(t) \]

where \( g \) is the forcing term (maybe other cells, etc.). Assume that \( f(x) > 0 \) as usual and suppose that \( u(0) = 0 \) and \( u(T) = 1 \) is the “periodic” solution. Since \( u(t) \) is monotone, then introduce a new variable, \( \theta \) defined by \( x = u(\theta) \). Then \( x' = u'(\theta)\theta' \) so that we get

\[ u'(\theta)\theta' = f(u(\theta)) + g \]

since \( u' = f(u) \), we divide by \( u' \) and obtain:

\[ \frac{d\theta}{dt} = 1 + g/u'(\theta) = 1 + R(\theta)g(t) \]

where \( R = 1/u' \) is the PRC or adjoint. Thus, without making any a priori assumptions about oscillators, you can use an experimentally determined PRC to create a network of pulse coupling:

\[ \frac{d\theta_j}{dt} = 1 + R(\theta_j) \sum_k s_{jk}(t) \]

where \( s_{jk}(t) \) is the interaction from oscillator \( k \) to \( j \). \( R \) is an experimentally determined adjoint. Note that for functions \( R \) derived from circle models, \( R \) is always positive. However, many experimental PRCs have a negative component, so that one can regard this equation more generally than just having come from a circle model.
3 Two-dimensional models.

Consider

\[ x' = F(x, y) \quad y' = G(x, y) \]

which is a planar system. Suppose it has a limit cycle \((u, v)\). We want to find the adjoint. Since \(u^*u' + v^*v' = 1\), we could in principle reduce the solution to the adjoint problem to a one-dimensional linear equation with periodic coefficients. But rather than do that, I will consider two examples for which the adjoint can be explicitly found. Both are important and in some sense “generic” near certain bifurcations.

3.1 \(\lambda - \omega\) models

A simple class of planar ODEs has the form

\[ x' = f(r)x - g(r)y, \quad y' = f(r)y + g(r)x \]

where \(r = \sqrt{x^2 + y^2}\), \(f(r_0) = 0\), with \(r_0 > 0\), \(f'(r_0) < 0\) and \(g(r_0) > 0\). Under these assumptions, there is a stable periodic solution \((u, v) = (r_0 \cos \omega t, r_0 \sin \omega t)\) where \(\omega = g(r_0)\). We point out that the normal form for a supercritical Hopf bifurcation has \(g(r) = 1 + qr^2\) and \(f(r) = 1 - r^2\). This is easier to see if we put the equations into polar coordinates, \(x = r \cos \theta\), and \(y = r \sin \theta\),

\[ r' = rf(r), \quad r\theta' = rg(r). \]

Given we have a closed form solution, we could linearize the equations and compute the adjoint (maybe). Instead, I will compute the adjoint in a more indirect fashion. Consider:

\[ x' = f(r)x - g(r)y + \epsilon N_x, \quad y' = f(r)y + g(r)x + \epsilon N_y \]

where \(N_x, N_y\) are perturbations (that is, the coupling terms). We know that the phase evolves according to the average over a period of \(W^T(t)(N_x, N_y)\) where \(W(t)\) is the adjoint. Thus, if we can figure out the phase behavior in terms of integrals of something over \((N_x, N_y)\), then the coefficients of that “something” are the adjoint. Converting the perturbed system to polar coordinates, we get

\[ r' = rf(r) + \epsilon [N_x \cos \theta + N_y \sin \theta] \tag{4} \]
\[ \theta' = g(r) + \epsilon \frac{1}{r}[N_y \cos \theta - N_x \sin \theta] \tag{5} \]

Let’s write \(r = r_0 + \epsilon r_1\). \(r_1\) will vary only on the order of \(ct\) so that \(\epsilon r_1'\) will be of order \(\epsilon^2\). Thus, to lowest order, equation (4) implies

\[ r_0 f'(r_0) r_1 + N_x \cos \theta + N_y \sin \theta \]

allowing us to solve for \(r_1\)

\[ r_1 = -\frac{1}{r_0 f'(r_0)} (N_x \cos \theta + N_y \sin \theta). \]
Figure 2: Relaxation oscillation orbit $U(t)$ in the limit as $\mu \to 0$ showing phase-plane and time plots.

We plug this into equation (5) to obtain:

$$\theta' = \omega + \epsilon \left( f'(r_0) r_1 + \frac{1}{r_0} [N_x \cos \theta - N_y \sin \theta] \right)$$

Now, write $\theta = \omega t + \Phi$ and we get

$$\Phi' = \epsilon \frac{1}{r_0} [-\sin \omega t + \Phi - \frac{g'(r_0)}{f'(r_0)} \cos \omega t + \Phi] N_x$$

$$+ \frac{\cos \omega t + \Phi - \frac{g'(r_0)}{f'(r_0)} \sin \omega t + \Phi} {N_y}.$$  

So there we go! We can write

$$\Phi' = \epsilon W^T(t + \Phi)(N_x, N_y)$$

where

$$W(t) = \frac{1}{r_0 \omega} \begin{pmatrix} -\sin \omega t - \frac{g'(r_0)}{f'(r_0)} \cos \omega t \\ \cos \omega t - \frac{g'(r_0)}{f'(r_0)} \sin \omega t \end{pmatrix}$$

The extra factor of $1/\omega$ comes from the fact that our zero order solution to $\Phi$ is $\Phi' = \omega$ so that $\Phi$ is dimensionless. However, the, general theory has $\Phi$ as a time-like variable with $\Phi' = 1$, so we must divide by $\omega$. As a check, note that $W^T(u', v') = 1$ where $u = r_0 \cos \omega t, v = r_0 \sin \omega t$. Note also that for the Hopf bifurcation case, the adjoint is

$$W(t) = \frac{1}{1 + q} \begin{pmatrix} -\sin(1 + q)t + q \cos(1 + q)t \\ \cos(1 + q)t + q \sin(1 + q)t \end{pmatrix}$$
4 Relaxation oscillations

Consider the relaxation oscillation
\[
\begin{align*}
\mu x' &= f(x, y) \\
y' &= g(x, y)
\end{align*}
\]
where \(0 < \mu \ll 1\). Let \(U(t)\) be the singular orbit (two pieces with two jumps).

The adjoint equation along with a normalization is
\[
\begin{align*}
u_t^* &= -(f_x/\mu)u^* - g_x v^* \\
v_t^* &= -(f_y/\mu)u^* - g_y v^* \\
1 &= (f/\mu)u^* + gv^*
\end{align*}
\]
Let \(w^* = u^*/\mu\) be a rescaling. Then the above becomes
\[
\begin{align*}
\mu w_t^* &= -f_xw^* - g_x v^* \\
v_t^* &= -f_yw^* - g_y v^* \\
1 &= fw^* + gv^*
\end{align*}
\]
Now, let \(\mu\) go to zero. Note that on the singular orbit and away from the jumps \(f = 0\), so that the normalization condition becomes
\[
v^* = 1/g
\]
and the first equation implies that
\[
w^* = -(g_x/f_x)v^* = -g_x/(f_xg) \equiv w_0^*.
\]
Thus, away from the jumps, we have \((w^*, v^*)\). Consider the jump at \(t_1\). Then \(v^*\) jumps from \(a_1\) to \(b_1\) (see the figure). We integrate the middle equation across this jump:
\[
\int_{t_1^-}^{t_1^+} v_t^* dt = \frac{1}{g(b_1)} - \frac{1}{g(a_1)} = -\int_{t_1^-}^{t_1^+} f_yw^*(t) dt - \int_{t_1^-}^{t_1^+} g_yv^*(t) dt
\]
The last integral is zero since \(g_y\) and \(v^*\) are integrable (having only finitely many discontinuities). The first integral right-hand side is not zero since we expect \(w^*\) to behave like a delta function. Letting \(w^* = w_0^* + c_1\delta(t - t_1)\) and integrating, we get that
\[
c_1 = \left(\frac{1}{f_y(a_1)} - \frac{1}{g(a_1)}\right)/ \left(\frac{1}{g(b_1)} - \frac{1}{g(a_1)}\right)
\]
A similar calculation across the second jump yields
\[
c_2 = \left(\frac{1}{f_y(a_2)} - \frac{1}{g(a_2)}\right)/ \left(\frac{1}{g(b_2)} - \frac{1}{g(a_2)}\right)
\]
so that
\[
    w^*(t) = -g_x/(f_x g) + c_1 \delta(t - t_1) + c_2 \delta(t - t_2)
\]
and \( u^* = \mu w^* \). So, we see that for a relaxation oscillation, \( u^* \) is \( O(\mu) \). This may seem odd, but it turns out to be exactly what we would like in the coupled case.

Consider:
\[
    \mu x' = f(x, y) + \epsilon P(U(t), U(t + \psi))
\]
\[
    y' = g(x, y) + \epsilon Q(U(t), U(t + \psi))
\]

Dividing by \( \mu \) in the first equation and applying averaging, we see that
\[
    H(\psi) = \frac{1}{T} \int_0^T w^*(t)P(U(t), U(t + \psi)) + v^*(t)Q(U(t), U(t + \psi)) \, dt
\]
That is, the \( \mu \)'s cancel out! Part of \( H(\psi) \) has delta functions. Look at for example
\[
    \int_0^T \delta(t - t_1)P(U(t), U(t + \psi)) \, dt.
\]
There will be discontinuities at places where \( U(t_1 + \psi) \) has a jump (e.g. at \( \psi = 0 \) and at \( \psi = t_2 - t_1 \)). This leads to interesting phase dynamics and synchrony can be ill-defined.

5 Numerical Methods.

It should be fairly obvious that it is not easy to compute the adjoint in general and thus one needs to apply numerical techniques. Let \( A(t) \) be the Jacobi matrix evaluated around a stable limit cycle to \( U^* = F(U) \), that is \( A(t) = D_X F(U(t)) \).

Then we know that
\[
    Y^* = A(t)Y
\]
has a unique (up to scalar multiplication) periodic solution, \( U'(t) \). We can write the general solution to this linear problem as
\[
    Y(t) = c_1 U'(t) + \sum_{k>1} c_k e^{a_k t} P_k(t)
\]
where the real parts of \( a_k \) are negative and \( P_k \) ai periodic. (This follows from Floquet theory.) Each of these is a solution to the original equation. From the homework, we know that any solution to the adjoint equation \( X' = -A^T(t)X \) satisfies \( X^TY = C \) where \( C \) is a constant. Consider the nonperiodic solutions that have the form \( \exp(a_k t)P_k(t) \). These exponentially decay so that solutions to the adjoint which are not orthogonal to \( P_k \) must exponentially grow. Thus, we can expect that in general solutions to the adjoint must exponentially grow except for the unique periodic solution. This suggests a numerical scheme for computing the adjoint. Integrate the equation
\[
    \frac{dX}{dt} = -A^T(t)X
\]
backwards in time starting with some random initial condition. (Unless you are very unlucky, this will always include some component of the periodic solution.) Since we are integrating backwards, all the terms of the adjoint which grow in forward time will decay away and the solution will converge to a periodic orbit. Then just normalize this with respect to \( U'(t) \) and you have the adjoint.

XPP does this for you once you have computed a limit cycle. Here are the steps to use it:

1. Compute a limit cycle for several cycles to make sure you are right on it.
2. Adjust the total amount of time to be as close as possible to the period of the limit cycle by changing Total in the Numerics menu.
3. If you want to place the time origin at a particular event, say, the peak of the action potential (which improves XPP’s accuracy at getting the adjoint), open the Data Browser and scroll until the peak of the AP is in the top row. In the Browser, click on Get which grabs this as an initial condition. Integrate one more time. Now you have a full orbit with the peak at \( t = 0 \).
4. In the Numerics menu, click on Bounds and change it to a big number, say 1000000. Click on Averaging New adjoint. Click Escape and plot the variable whose adjoint you desire. It should be there.

Here is an example of the little sodium-potassium model all set up for one period.

\[
\begin{align*}
  v' &= (I - g_l (V - e_l) - g_k n (v - e_k) - g_{na} \minf(v)(v - e_{na})) / C \\
  n' &= (\minf(v) - n) / \tau(v) \\
  \minf(v) &= 1 / (1 + \exp(-(v - va) / vb)) \\
  \ninf(v) &= 1 / (1 + \exp(-(v - vc) / vd)) \\
  \tau(v) &= \tau_0 \\
  \text{par I} &= 40, \text{par} \text{vc} = -45, \text{tau}0 = 1, \text{gl} = 8, \text{gk} = 10, \text{gna} = 20, \text{el} = -78, \text{ek} = -90, \text{ena} = 60 \\
  \text{par va} &= -20, \text{vb} = 15, \text{vd} = 5 \\
  \text{par c} &= 1 \\
  @ \text{xp} &= t, \text{yp} = v, \text{xlo} = 0, \text{xhi} = 4, \text{ylo} = -80, \text{yhi} = 0 \\
  @ \text{bound} &= 1000, \text{dt} = .01, \text{meth} = \text{qualrk}, \text{total} = 3.66 \\
  \text{init} v &= -8.1692323, n = .510050 \\
  \text{done}
\end{align*}
\]

6 HOMEWORK

1. Show that \( L^t Y = -dY/dt - A^t(t)Y \).

2. Suppose that \( dY/dt = A(t)Y \) and \( -dX/dt = A^t(t)X \). Show that

\[
X^t(t)Y(t) = C,
\]

where \( C \) is an arbitrary constant. (Hint: differentiate \( X^t(t)Y(t) \).)
Figure 3: Adjoint for simple sodium-potassium model when $I = 40$.

3. Let $x' = f(x)$ be an equation on the circle $x \in [0, 1)$ with $f(x) > 0$. Let $u(t)$ be the solution to $u' = f(u)$ with $u(0) = 0$ and let $T$ be the value of $t$ such that $u(T) = 1$. The adjoint to the linearized equation satisfies:

$$-y' = f_x(u(t))y$$

show that $y(t) = 1/u'(t)$.

4. Consider the quadratic integrate-and-fire model on the real line:

$$x' = x^2 + a^2$$

where $a > 0$. The “periodic” solution to this satisfies $\lim_{t \to 0} u(t) = -\infty$ and $\lim_{t \to T} u(t) = \infty$. First find $u(t)$ where

$$\int_{-\infty}^{u(t)} \frac{dx}{x^2 + a^2} = t$$

What is the period $T(a)$? The solution to the adjoint is $u^*(t) = 1/u'(t)$. Compute this adjoint explicitly and thus show a fundamental result about adjoints and Type I excitable systems.

5. In the QIF model, with a synapse $s(t)$, compute the coefficients, $A, B, C$ in equation (3) in terms of integrals of $s(t)$.

6. Consider a weakly coupled circle model:

$$x'_j = f(x_j) + \epsilon s_k(t)$$
where \( s_k(t) = \exp(-t/\tau) \) is the cell \( k \) fires at \( t = 0 \). That is the synapse is instantly rising and decays monotonically. Assume that \( f(x) > 0 \) so that the uncoupled system is an oscillator. Show that no matter what the function \( f(x) \), synchrony is always an unstable solution to the weakly symmetrically coupled system for \( \epsilon > 0 \) and a stable solution when \( \epsilon < 0 \) (inhibitory coupling). What can you say about the out-of-phase solution where the two oscillators are \( T/2 \) apart?

7. Prove that for \( T \)-periodic functions, \( U, V \) that

\[
\int_0^T U^T(t)V(t + \psi) \, dt = \int_0^T U^T(t - \psi)V(t) \, dt.
\]

Use this to compute:

\[
H(\psi) = \frac{1}{T} \int_0^T (1 - \cos(2\pi t/T))s(t + \psi) \, dt
\]

where \( s(t) = \hat{s}(t) = \exp(-t/\tau) \) for \( t < T \) and \( s(t) = \hat{s}(t - T) \) for \( T \leq t < 2T \), etc. Compute \( g(\psi) = H(-\psi) - H(\psi) \) and find all the fixed points and their stability.

8. Find values of \( a, b \) such that synchrony is a stable solution to the QIF above when \( s(t) = (\exp(-at) - \exp(-bt))/(b - a) \). (Hint: rather than compute \( H(\psi) \), just compute \( H'(0) \).)

9. Consider the following coupled system:

\[
\frac{dX_j}{dt} = F(X_j(t)) + \epsilon G(X_j(t), X_k(t - \tau))
\]

which is a delayed coupling between the two oscillators. Show that

\[
H_\tau(\psi) = H(\psi - \tau)
\]

where \( H(\psi) \) is the interaction for no delay and \( H_\tau \) is the interaction with delay. In general suppose the coupling is

\[
G(X_j(t), Y_k(t))
\]

where

\[
Y(t) = \int_0^\infty M(t')X(t - t') \, dt'.
\]

Note that this is history-dependent coupling. So, if \( M(t) = \delta(t - \tau) \), then we recover the delayed coupling. Show that

\[
H_M(\psi) = \int_0^\infty M(t')H(\psi - t') \, dt'
\]

where \( H_M \) is the interaction with this history dependence and \( H \) is the direct interaction function.
10. Show that if \( G(U_j, U_k) = D(U_k - U_j) \) where \( D \) is a constant matrix, then \( H(0) = 0 \). Show that synchrony is always a stable solution to scalar diffusive coupling:

\[
G(U_j, U_k) = U_k - U_j.
\]

11. Compute \( H(\psi) \) for the system

\[
x'_j = f(r_j)x_j - g(r_j)y_j + \varepsilon d_x x_k, \quad y'_j = f(r_j)y_j + g(r_j)x_j + \varepsilon d_y y_k
\]

where \( r_j = \sqrt{x_j^2 + y_j^2} \), \( f(r) = 1 - r^2 \) and \( g(r) = 1 + q r^2 \). You should use the adjoint that was already derived for you. What are all the possible phase-locked solutions and what is their stability?