Homework # 6

1. An activator-inhibitor system at a Hopf bifurcation has linearization:

\[ M = \begin{pmatrix} a & -b \\ c & -a \end{pmatrix} \]

where \( a, b, c \) are positive parameters. Suppose that \( bc > a^2 \). Compute the eigenvector, \( \Phi \) corresponding to the imaginary eigenvalue, \( i\omega = i\sqrt{bc - a^2} \). Compute the adjoint eigenvector \( \Psi \) where \( M^T \Psi = -i\omega \Psi \) and \( \bar{\Psi}^T \Phi = 1 \).

The coupling matrix is:

\[ C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

Write down the quantity, \( \beta = \bar{\Psi}^T C \Phi \) and show that \( \beta \) has a positive real part.

2. Recall that a pair of coupled Hopf bifurcations satisfies:

\[
\begin{align*}
z_1' &= z_1(1 + (-1 + iq)|z_1|^2) + (\mu + i\gamma)(z_2 - z_1) \\
z_2' &= z_2(1 + (-1 + iq)|z_2|^2) + (\mu + i\gamma)(z_1 - z_2)
\end{align*}
\]

Let \( z_j = r_j \exp(i\theta_j) \) and let \( \phi = \theta_2 - \theta_1 \). Show that

\[
\begin{align*}
r_1' &= r_1 - r_1^3 + \mu(r_2 \cos \phi - r_1) - \gamma r_2 \sin \phi \\
r_2' &= r_2 - r_2^3 + \mu(r_1 \cos \phi - r_2) + \gamma r_1 \sin \phi \\
\phi' &= q(r_2^2 - r_1^2) - \mu \left( \frac{r_2}{r_1} + \frac{r_1}{r_2} \right) \sin \phi + \gamma \left( \frac{r_1}{r_2} - \frac{r_2}{r_1} \right) \cos \phi
\end{align*}
\]

Find conditions for which there is a solution of the form, \( r_1 = r_2 = R \) and \( \phi = \pi \). Suppose that \( q = 2 \) and \( \gamma = 1 \). Use the computer to explore the behavior as a function of \( \mu \) starting with \( \mu \) near zero and going up to 1.

3. Consider a ring of coupled Hopf bifurcations:

\[
z_j' = z_j[1 + (-1 + iq)|z_j|^2] + (\mu + i\gamma)(z_{j+1} - 2z_j + z_{j-1}), \quad j = 1, \ldots, N
\]

with \( z_0 \equiv z_N \) and \( z_{N+1} \equiv z_1 \). Find conditions for which there exists a traveling wave, \( z_j = R_N \exp(i\Omega t + 2\pi j/N) \) and find expressions for \( R_N, \Omega_N \).

4. Consider the chain:

\[
\begin{align*}
\theta_1' &= \omega + \sin(\theta_2 - \theta_1 - \alpha) \\
\theta_j' &= \omega + \sin(\theta_{j+1} - \theta_j - \alpha) + \sin(\theta_{j-1} - \theta_j + \alpha) \\
\theta_N' &= \omega + \sin(\theta_{N-1} - \theta_N + \alpha)
\end{align*}
\]

Find a traveling wave for this, \( \theta_j = \Omega t + cj \) and determine it’s stability.
5. Consider the weakly coupled pair of oscillators:

\[
\begin{align*}
\theta'_1 &= \omega + \epsilon P(\theta_2) \Delta(\theta_1) \\
\theta'_2 &= \omega + \epsilon P(\theta_1) \Delta(\theta_2)
\end{align*}
\]

Let \( \theta_j = \psi_j + \omega t \) and note that

\[
\begin{align*}
\psi'_1 &= \epsilon P(\omega t + \psi_2) \Delta(\omega t + \psi_1) \\
\psi'_2 &= \epsilon P(\omega t + \psi_1) \Delta(\omega t + \psi_2)
\end{align*}
\]

which we can average, yielding

\[
\begin{align*}
\psi'_1 &= \epsilon H(\psi_2 - \psi_1) \\
\psi'_2 &= \epsilon H(\psi_2 - \psi_2)
\end{align*}
\]

where \( H(\phi) = \int_0^1 P(s + \phi) \Delta(s) ds \).

Letting \( \phi = \psi_2 - \psi_1 \), we get

\[
\phi' = H(-\phi) - H(\phi)
\]

Suppose that \( P(\theta) = 0 \) for \( |\theta - \theta_T| > \sigma \) and \( P(\theta) = 1/(2\sigma) \) for \( |\theta - \theta_T| < \sigma \) (of course modulo 1, so that if \( \theta_T = 0 \), then \( P \) vanishes in the interval \((\sigma, 1 - \sigma))\). Assume that \( \sigma < 1/2 \). Determine all the phase-locked states (equilibria of the \( \phi \) equation), when \( \Delta(\theta) = -\sin 2\pi \theta \) and \( \Delta(\theta) = 1 - \cos 2\pi \theta \). Note that these correspond respectively to the adjoints near a Hopf and a saddle-node.

6. Obtain the complete bifurcation diagram for the self-coupled excitatory neuron model without saturation:

\[
\frac{ds}{dt} = \alpha_0 \sqrt{g(s-s^*)} - s
\]

where the term inside the square root is set to zero if \( s < s^* \). Assume, \( s^*, \alpha_0, g \) are all positive.