Homework #3

1. The quadratic integrate and fire model can have squarewave bursts in some parameter regimes. Consider:

\[ V' = V^2 + a - W, \quad W' = \epsilon W \]

where when \( V = 10 \), \( V \) is reset to 1 and \( W \) is incremented by \( d \). For what range of \((a, d)\) does fast-slow analysis predict bursting. Here is how to proceed. Treat \( W \) as a parameter and look at the \( V \) equation. Figure out where the "homoclinic" is and when there will be repetitive spiking. Figure out how big \( W \) has to get to stop the spiking. Then you can estimate how long it will take \( W \) to decay enough to start spiking again. Use this to get an estimate of \( a \). Note that \( d \) will mostly control how many spikes per burst. If you set \( \epsilon \) small enough, you can pretend that \( W \) does not decay at all between spikes and from this figure out how many spikes per burst as a function of \( a, d \). You may want to simulate this to check your theory.

2. As I mentioned in class, you can create a 1-dimensional map for the burster. The map is piecewise defined and I have found it to be:

\[
\begin{align*}
f_1(x) &= a_1 + b_1 x \\
f_2(x) &= f_1(x_1) - b_2(x-x_1) \\
f_3(x) &= f_2(x_2) \\
a_1 &= 0.2, b_1 = 0.8, b_2 = 15, x_1 = 1.3, x_2 = 1.36 \\
f(x) &= \text{if}(x < x_1) \text{then}(f_1(x)) \text{else}(\text{if}(x < x_2) \text{then}(f_2(x)) \text{else}(f_3(x)))
\end{align*}
\]

The map is defined as:

\[ x_{n+1} = f(x_n) + i \]

where \( i \) is the parameter. A fixed point is defined as \( x = f(x) \) and it will be stable if \( |f'(x)| < 1 \). A fixed point corresponds to a tonic spiking solution in the burster if it occurs on the first part of \( f \). That is, it is not bursting. Find the maximum value of \( i \) so that there is a tonic spiking solution. Note that the slope of \( f \) is 0.8 on the first part, \(-15\) on the second part, and 0 on the third part. You should be able to do this analytically since you just have to solve a linear equation. Through simulation, try to find some periodic solutions, that is, \( x_{n+M} = x_n \) for \( M > 1 \). For example, if you pick \( i = 0.15 \), you should find a period 11 solution! Try to get period 5, 4, and 3. At what value of \( i \) does spiking cease? Hint: this corresponds to a fixed point on the flat part of \( f \) \((x > x_2)\).

3. For the elliptic burster, we consider the simple model. If we ignore the \( \theta \) variable, then:

\[
\begin{align*}
r' &= r(p + r - r^2) \\
p' &= \epsilon(r_0 - r)
\end{align*}
\]
Simulate this simple model for \( r_0 = 0.5 \) and \( \epsilon = .05 \). You will see an oscillation in \( r \) if you start with initial conditions, \( r(0) > 0 \) and, say, \( p = 0 \). This corresponds to bursting since the regions where \( r \) is near zero are silent and those where \( r \) is close to 1 are spiking. Regular spiking corresponds to an equilibrium where \( r > 0 \). Find this equilibrium and its stability. For what values of \( \epsilon, r_0 \) will there be a Hopf bifurcation. Compute the period in the singular limit; it involves computing this integral:

\[
f(r_0) = \int_0^{-1/4} dp / (r_0 - (1/2 + \sqrt{(1/4 + p)}))
\]

(Maple make a mess of this, but it is really not so bad. REDUCE works much better!) If you want, try to compute the singular period in terms of \( r_0 \) given this integral. (The key here is showing that this integral actually shows up!)

4. Now onto parabolic bursting. Consider:

\[
\begin{align*}
\dot{u} &= 1 - \cos(u) + (1 + \cos(u))(a + b \cdot x - c \cdot y) \\
\dot{x} &= \epsilon(-x + \delta(u - \pi)) \\
\dot{y} &= \epsilon(-y + \delta(u - \pi)) / \tau
\end{align*}
\]

where by \( \delta(u - \pi) \) we mean that each time \( u \) crosses \( \pi \), increment by 1. If you fix \( x, y \), show that the frequency of \( u \) (That is, the period is the time it take \( u \) to go from \(-\pi \) to \( \pi \) is \( 1/f \)) is \( f = \sqrt{\max(a + bx - cy, 0)} \). The average of \( \delta(u - \pi) \) is exactly \( f \). Thus, the averaged \((x, y)\) system is:

\[
\begin{align*}
\dot{x} &= \epsilon(-x + f) \\
\dot{y} &= \epsilon(-y + f) / \tau
\end{align*}
\]

Show via nullclines, simulation, etc, that there are some values of \( a, b, c, \tau \) where the above system has a limit cycle. Note by rescaling time in the above, you can set \( \epsilon = 1 \).

5. Compute the velocity of the wavefront in the piecewise linear model:

\[
V_t = f(V) + V_{xx}
\]

where \( f(V) = -V + H(V - \theta) \) and \( H \) is the step function. Proceed as follows. First the traveling system is

\[
-\epsilon V' = f(V) + V''
\]

Note that \( f(0) = f(1) = 0 \) when \( 0 < \theta < 1 \). You want a solution that satisfies \( V(-\infty) = 1 \) and \( V(\infty) = 0 \). Note that the equation is always linear except at the jump. Finally, since the wave is translation invariant, you should choose coordinates so that \( V(0) = \theta \). So, \( V(\xi) > \theta \) for \( \xi < 0 \) and \( V(\xi) < \theta \) for \( \xi > 0 \).
6. Using the results of the above exercise, compute the singular solution to the Rinzel pulse model:

\[-cV' = f(V) - w + V''\]
\[-cw' = \epsilon[V - bw]\]

Note that you should make sure $b$ is chosen so that there is only the equilibrium $(0, 0)$. Proceed as follows. When $\epsilon = 0$ start at rest, so $w = 0$ and use the previous exercise to jump up to the $V > \theta$ branch. Now rescale $\xi$ and set $\epsilon = 0$ to get

\[0 = f(V) - w\]
\[-cw' = V^+(w) - bw\]

where you need to solve for $V^+(w)$. This is a linear equation in $w$ starting with $w(0) = 0$ so you have to integrate it until $w = w_{\text{jump}}$. Figure out $w_{\text{jump}}$ from the previous problem since it requires that the velocity be opposite the jump up velocity. Last but not least, compute the solution to the $w'$ equation with $V^+$ replaced by $V^-$, the low $V$ root of $f(V) - w = 0$. An interesting extension of this is to try to compute the singular periodic orbits that happen for a lower value of $c$ than the singular homoclinic. Here is how to do this. Pick a small value of $w$, call it $w_1$ that is near zero. There will be 2 roots to $f(V) - w_1 = 0$, so choose $c(w_1)$ to create a wave that jumps from one to the other. Now on the right branch $V^+(w)$, solve the $w$ system. Figure out the place to jump back ($w_2$) to the $V^-$ side by matching the velocity to your jump up velocity. Then solve the $w$ equation on $V^-$ until $w$ again equals $w_1$. The total transit time is the period, $T$. Both $c$ and $T$ are parametrized by $w_1$, so you should be able to compute the singular dispersion relation!

7. Problem 1,6,7 Chapter 6.