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Bifurcations of traveling wave solutions in a coupled non-linear wave equation

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Abstract

By using the theory of planar dynamical systems to a coupled non-linear wave equation, the existence of solitary wave solutions and uncountably infinite, many smooth and non-smooth, periodic wave solutions is obtained. Under different parametric conditions, various sufficient conditions to guarantee the existence of the above solutions are given.

1. Introduction

In 1985, Kupershmidt [7] posed a multicomponent Korteweg-de Vries equation with dispersion. It reads as

$$u_t = -u_{xxx} + 6uu_x + 2\mathbf{v}^{\mathrm{T}}\mathbf{v}_x + \mathbf{c}^{\mathrm{T}}\mathbf{v}_{xx}, \quad \mathbf{v}_t = (2u\mathbf{v})_x - \mathbf{c}u_{xx}, \tag{1.1}$$

where

$$\mathbf{v} = (v_1, v_2, \dots, v_n)^{\mathrm{T}}, \quad \mathbf{c} = (c_1, c_2, \dots, c_n)^{\mathrm{T}}, \tag{1.2}$$

c_i is a constant. For $\mathbf{c} \neq 0$, Kwek and Li [6] have considered the bifurcations of smooth and non-smooth travelling wave solutions of (1.1). When $n = 1$ and $\mathbf{c} = 0$, (1.1) is reduced to the following coupled non-linear equation:

$$u_t = -u_{xxx} + 6uu_x + 2vv_x, \quad v_t = 2(uv)_x. \tag{1.3}$$

The transformation $u \to -u$, $t \to -t$ makes (1.3) become

$$u_t = u_{xxx} + 6uu_x + 2vv_x, \quad v_t = 2(uv)_x. \tag{1.4}$$

Eq. (1.4) were proposed to describe the interaction process of two internal long waves. In [4], Ito presented a recursion operator by which he inferred that Eq. (1.4) possess infinitely many symmetrics and constants of motion. Guo and Tan [3] studied the existence of global smooth solution of initial-value problem for (1.4).

In this paper we shall consider bifurcation behaviour of the traveling wave solutions for more general equation:

$$u_t = u_{xxx} + buu_x + 2vv_x, \quad v_t = 2(uv)_x, \tag{1.5}$$

where $b \neq 0$ and when $b = 6$, (1.5) just is (1.4).

Let $u(x,t) = \phi(x - ct) = \phi(\xi)$, $v(x,t) = v(x - ct) = v(\xi)$, where c is the wave speed.

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Substituting the above traveling wave solutions into (1.5), we have

$$-c\phi' = \phi''' + \frac{b}{2}(\phi^2)' + (v^2)', \quad -cv' = (2\phi v)', \tag{1.6}$$

where "′" is the derivative with respect to ξ. Integrating (1.6) with respect to ξ, we obtain

$$\phi'' = -c\phi - \frac{b}{2}\phi^2 - v^2, \quad -cv = (2\phi v) + g, \tag{1.7}$$

where g is an integral constant. We suppose that $g \neq 0$, otherwise we only have the trivial solution. Thus, we see from the second equation of (1.7) that

$$v = \frac{-g}{c + 2\phi}. \tag{1.8}$$

Substituting (1.8) into the first equation of (1.7), it gives rise to the following traveling wave equation of (1.5):

$$\phi'' + c\phi + \frac{b}{2}\phi^2 + \frac{g^2}{(c+2\phi)^2} = 0. \tag{1.9}$$

The Eq. (1.9) is equivalent to the two-dimensional system as follows

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -c\phi - \frac{b}{2}\phi^2 - \frac{g^2}{(c+2\phi)^2} \tag{1.10}$$

which has the first integral (Hamiltonian)

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{c}{2}\phi^2 + \frac{b}{6}\phi^3 - \frac{g^2}{2(c+2\phi)}. \tag{1.11}$$

System (1.10) is a three-parameter planar dynamical system depending on the parameter group (c, g, b). Because of the phase orbits defined by the vector fields of system (1.10) determine all travelling wave solutions of Eq. (1.5). We shall investigate the bifurcations of phase portraits of (1.10) in the phase plane (ϕ, y) as the parameters c, g and b are changed.

We point out that here we are considering a physical model where only bounded traveling waves are meaningful. So that we only pay attention to the bounded solutions of (1.10).

Suppose that $u(x,t) = \phi(x - ct) = \phi(\xi)$ is a continuous solution of Eq. (1.5) for $\xi \in (-\infty, \infty)$ and $\lim_{\xi\to\infty} \phi(\xi) = \alpha$, $\lim_{\xi\to-\infty} \phi(\xi) = \beta$. It is well known that (i) $u(x,t)$ is called a solitary wave solution if $\alpha = \beta$. (ii) $u(x,t)$ is called a kink or antikink solution if $\alpha \neq \beta$. Usually, a solitary wave solution of (1.5) corresponds to a homoclinic orbit of (1.10). A kink (or antikink) wave solution (1.5) corresponds to a heteroclinic orbit (or so called connecting orbit) of (1.10). Similarly, a periodic orbit of (1.10) corresponds to a periodically traveling wave solution of (1.5). Thus, to investigate all bifurcations of solitary waves, kink waves and periodic waves of Eq. (1.5), we shall find all periodic annuli, homoclinic and heteroclinic orbits of (1.10) depending on the parameter space (c, g, b) of this system. The bifurcation theory of dynamical systems (see [1,5,12]) plays an important role in our study.

We notice that the right hand of the second equation in (1.10) is not continuous when $\phi = -c/2$. In other words, on the above straight line of the phase plane (ϕ, y), ϕ''_ξ has no definition. It implies that a smooth system (1.5) sometimes has non-smooth traveling wave solutions. This phenomenon has been considered by some authors (see [6,8,10] and [11]). We claim that the existence of a singular straight line for a traveling wave equation is the original reason why traveling waves lose their smoothness (see [8]).

The paper is organized as follows. In Section 2, we discuss bifurcations of phase portraits of (1.10). Explicit parametric conditions will be given. In Section 3, we consider the existence of smooth solitary traveling wave and periodic traveling wave solutions of (1.5). In Section 4, we show the existence of non-smooth periodic traveling wave solutions.

2. Bifurcation set and phase portraits of (1.10)

Let $\psi = \phi + (c/2)$. The system (1.10) is reduced to

$$\frac{d\psi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\left(\frac{(2-b)c}{2}\psi + \frac{b}{2}\psi^2 - \frac{4-b}{8}c^2 + \frac{g^2}{4\psi^2}\right). \tag{2.1}$$

First, we suppose that $(2-b)c \neq 0$. To simplify our study, it is more convenient to use the transformation

$$\psi = \left|\frac{(2-b)c}{b}\right| q, \quad y = \left|\frac{(2-b)c}{b}\right| p, \quad \mathrm{d}\xi = \frac{2}{|(2-b)c|} q^2 \,\mathrm{d}\zeta.$$

Then the system (2.1) has the same topological phase portraits as the following quartic polynomial system

$$\frac{\mathrm{d}q}{\mathrm{d}\zeta} = \frac{2}{|(2-b)c|} q^2 p, \quad \frac{\mathrm{d}p}{\mathrm{d}\zeta} = -\mathrm{sign}(b) f(q), \tag{2.2}$$

except for the straight line $q = 0$, where

$$f(q) = q^4 + \mathrm{sign}\left(\frac{(2-b)c}{b}\right) q^3 - \beta q^2 + \alpha,$$

$$\alpha = \frac{b^3 g^2}{2c^4 (2-b)^4}, \quad \beta = \frac{1}{(2-b)^2} - \frac{1}{4}.$$

For the new system (2.2), $q = 0$ is an invariant straight line solution and it has the first integral as follows:

$$H(q,p) = p^2 + H_1(q) = h, \tag{2.3}$$

where h is an integral constant and

$$H_1(q) = \mathrm{sign}(b) \frac{|(2-b)c|}{3} \left(q^3 + \frac{3}{2} \mathrm{sign}\left(\frac{(2-b)c}{b}\right) q^2 - 3\beta q - 3\frac{\alpha}{q} \right). \tag{2.4}$$

Note that for a fixed h, (2.3) determines a set of invariant curves of (2.2), which contains two, three or four different branches of curves. As h is varied, (2.3) defines different families of orbits of (2.2) having different dynamical behaviour (see the discussion below).

To investigate the critical points of (2.2), we need to find all zeros of the function $f(q)$. Notice that

$$f'(q) = q\left(4q^2 + 3\mathrm{sign}\left(\frac{(2-b)c}{b}\right) q - 2\beta\right). \tag{2.5}$$

Clearly, $f'(q)$ has three real zeros at $q_0 = 0$ and

$$q_1 = \frac{1}{8}\left(-3\mathrm{sign}\left(\frac{(2-b)c}{b}\right) + \sqrt{9+32\beta} \right) = \frac{1}{8}\left(-3\mathrm{sign}\left(\frac{(2-b)c}{b}\right) + \sqrt{1 + \frac{32}{(2-b)^2}} \right),$$

$$q_2 = \frac{1}{8}\left(-3\mathrm{sign}\left(\frac{(2-b)c}{b}\right) - \sqrt{1 + \frac{32}{(2-b)^2}} \right).$$

The relations $f(q_1) = 0, \quad f(q_2) = 0$ imply the parameter conditions

$$a \equiv \frac{g^2}{c^4} = \frac{1}{256 b^3} (128 + 80(2-b)^2 - (2-b)^4 + |2-b|(32 + (2-b)^2)^{3/2}), \tag{2.6}$$

$$a \equiv \frac{g^2}{c^4} = \frac{1}{256 b^3} (128 + 80(2-b)^2 - (2-b)^4 - |2-b|(32 + (2-b)^2)^{3/2}). \tag{2.7}$$

Let $M(q_e, 0)$ be the coefficient matrix of the linearized system of (2.2) at an equilibrium point $(q_e, 0)$. At this equilibrium point we have

$$J(q_e, 0) = \det M(q_e, 0) = \frac{2}{|(2-b)c|} \mathrm{sign}(b) q_e^2 f'(q_e). \tag{2.8}$$

By the theory of planar dynamical systems (see [1,5,12]), for an equilibrium point of an planar Hamiltonean system, if $J < 0$, then the equilibrium point is a saddle point; If $J > 0$, then it is a center point; if $J = 0$ and the Poincaré index of the equilibrium point is 0, then this equilibrium point is a cusp.

1. The case $(2-b)c > 0$, i.e., $b > 2$, $c < 0$ or $b < 2$, $c > 0$. Notice that $a = (g^2/c^4) > 0$. In the upper (b, a)-parameter half-plane, there exist four bifurcation curves:

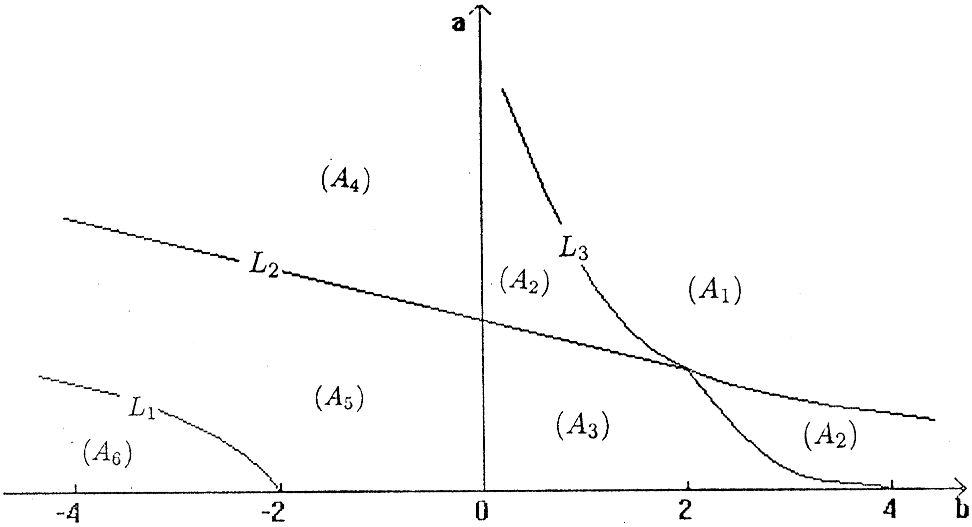

Fig. 1. Six regions in the upper (b,a)-parameter half-plane.

$$L_0: \ b=0; \quad L_1: a=-\frac{3}{512b^3}(16-(2-b)^2)^2, \ b<-2,$$

$$L_2: \ a=\frac{1}{256b^3}(128+80(2-b)^2-(2-b)^4-|2-b|(32+(2-b)^2)^{3/2}), \tag{2.9}$$

$$L_3: \ a=\frac{1}{256b^3}(128+80(2-b)^2-(2-b)^4+|2-b|(32+(2-b)^2)^{3/2}). \tag{2.10}$$

These curves partition the upper (b,a)-parameter half-plane into 6 regions denoted by $(A_1)-(A_6)$ shown in Fig. 1.

(i) For $(b,a)\in(A_1)$, there is not critical points of (2.2).
(ii) For $(b,a)\in L_3$, there exists a critical point of (2.2) which is cusp.
(iii) For $(b,a)\in(A_2)$, there exist two critical points of (2.2) at $(q_{e1},0)$ and $(q_{e2},0)$ with $q_{e1}<q_{e2}<0$. $(q_{e1},0)$ is a saddle, $(q_{e2},0)$ is a center. There is a homoclinic orbit to the saddle $(q_{e1},0)$, in which there exists a family of periodic orbits surrounding the center $(q_{e2},0)$. The phase portrait is shown in Fig. 2(a).
(iv) For $(b,a)\in L_2$, $b>0$, there exist three critical points of (2.2) at $(q_{ei},0)$, $i=1,2,3$ with $q_{e1}<q_{e2}<0<q_{e3}$. The point $(q_{e3},0)$ is a cusp. $(q_{e1},0)$ is a saddle, $(q_{e2},0)$ is a center. There is a homoclinic orbit to the saddle $(q_{e1},0)$, in which there exists a family of periodic orbits surrounding the center $(q_{e2},0)$. The phase portrait is shown in Fig. 2(b).
(v) For $(b,a)\in(A_3)$, there exist four critical points of (2.2) at $(q_{ei},0)$, $i=1,2,3,4$ with $q_{e1}<q_{e2}<0<q_{e3}<q_{e4}$. $(q_{e1},0)$ and $(q_{e3},0)$ are saddle points. $(q_{e2},0)$ and $(q_{e4},0)$ are centers. There are two homoclinic orbits to the saddle $(q_{e1},0)$ and $(q_{e3},0)$, respectively, in which there exist two families of periodic orbits surrounding the center $(q_{e2},0)$ and $(q_{e4},0,)$ respectively. The phase portrait is shown in Fig. 2(c).
(vi) For $(b,a)\in(A_4)$, there exist two critical points of (2.2) at $(q_{e1},0)$ and $(q_{e2},0)$ with $q_{e1}<0<q_{e2}$. $(q_{e1},0)$ is a center, $(q_{e2},0)$ is a saddle. There exists a family of periodic orbits surrounding the center $(q_{e1},0)$ The phase portrait is shown in Fig. 2(d).
(vii) For $(b,a)\in L_2$, $b<0$, there exist three critical points of (2.2) at $(q_{ei},0)$, $i=1,2,3$ with $q_{e1}<0<q_{e2}<q_{e3}$. The point $(q_{e2},0)$ is a cusp. $(q_{e3},0)$ is a saddle. There exists a family of periodic orbits surrounding the center $(q_{e1},0)$. The phase portrait is shown in Fig. 2(e).
(viii) For $(b,a)\in(A_5)$, there exist four critical points of (2.2) at $(q_{ei},0)$, $i=1,2,3,4$ with $q_{e1}<0<q_{e2}<q_{e3}<q_{e4}$. $(q_{e1},0)$ and $(q_{e3},0)$ are centers. $(q_{e2},0)$ and $(q_{e4},0)$ are saddle points. There is a homoclinic orbit to the saddle $(q_{e2},0)$, in which there exists a family of periodic orbits surrounding the center $(q_{e3},0)$. And there exists a family of periodic orbits surrounding the center $(q_{e1},0)$. The phase portrait is shown in Fig. 2(f).
(ix) For $(b,a)\in L_1$, there exist four critical points of (2.2) at $(q_{ei},0)$, $i=1,2,3,4$ with $q_{e1}<0<q_{e2}<q_{e3}<q_{e4}$. $(q_{e1},0)$ and $(q_{e3},0)$ are centers. $(q_{e2},0)$ and $(q_{e4},0)$ are saddle points. There are two heteroclinic orbits connecting

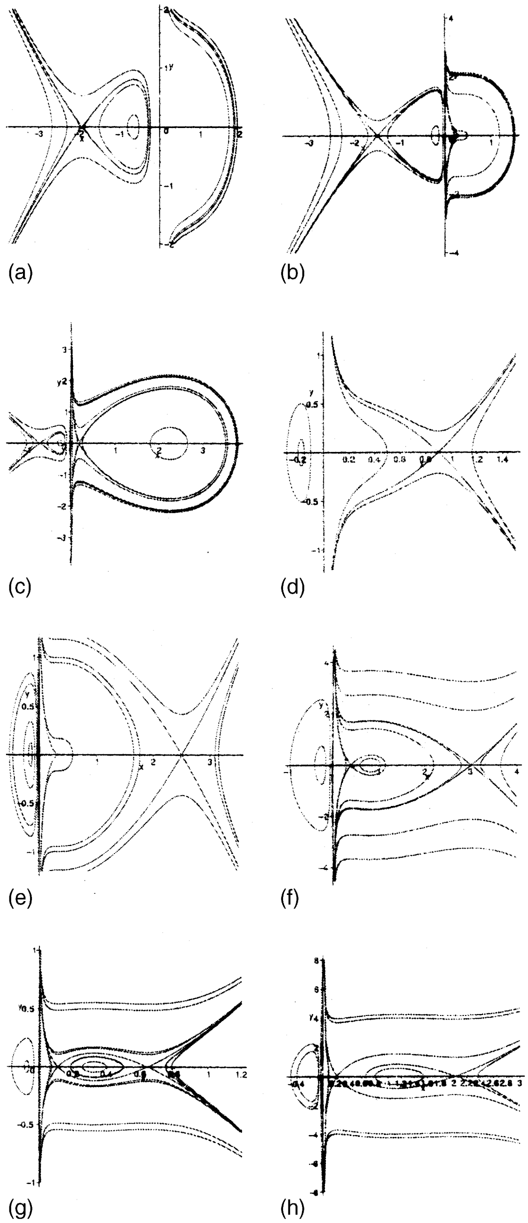

Fig. 2. Phase portraits of (2.2) when $(2-b)c>0$. (a) $(b,a)\in(A_2)$, $(c,g,b)=(1,1,0.8)$. (b) $(b,a)\in L_2, b>0, (c,g,b)=(-3,04519415489,3)$. (c) $(b,a)\in(A_3),(c,g,b)=(-1.5,,1)$. (d) $(b,a)\in(A_4)$, $(c,g,b)=(1,0.5,-6)$. (e) $(b,a)\in L_2$, $(c,g,b)=(1, 0.2835618945,-1)$. (f) $(b,a)\in(A_5), b>0, (c,g,b)=(2.4,1,-2)$. (g) $(b,a)\in L_1$, $(c,g,b)=(1,0.25,-6)$. (h) $(b,a)\in(A_6)$, $(c,g,b)=(2.4, 1,-6)$.

the saddle points $(q_{e2},0),(q_{e4},0)$, in which there exists a family of periodic orbits surrounding the center $(q_{e3},0)$. And there exists a family of periodic orbits surrounding the center $(q_{e1},0)$. The phase portrait is shown in Fig. 2(g).

(x) For $(b,a)\in(A_6)$, there exist four critical points of (2.2) at $(q_{ei},0)$, $i=1,2,3,4$ with $q_{e1}<0<q_{e2}<q_{e3}<q_{e4}$. $(q_{e1},0)$ and $(q_{e3},0)$ are centers. $(q_{e2},0)$ and $(q_{e4},0)$ are saddle points. There is a homoclinic orbit to the saddle $(q_{e4},0)$, surrounding the center $(q_{e3},0)$. And there exists a family of periodic orbits surrounding the center $(q_{e1},0)$. The phase portrait is shown in Fig. 2(h).

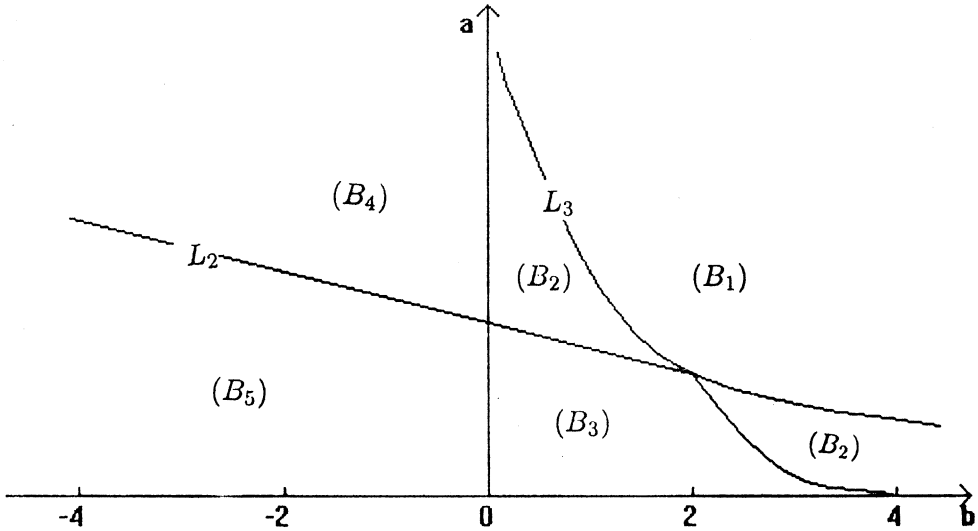

Fig. 3. Five regions in the upper (b, a)-parameter half-plane.

2. The case $(2-b)c<0$, i.e., $b>2, c<0$ or $b<2, c>0$. In this case, in the upper (b, a)-parameter half-plane, there exist three bifurcation curves: $L_0: \ b=0$, L_2 and L_3 defined by (2.9) and (2.10). These curves partition the upper (b, a)-parameter half-plane into five regions denoted by $(B_1)-(B_5)$ shown in Fig. 3.

(i) For $(b, a) \in (B_1)$, there is not critical points of (2.2).
(ii) For $(b, a) \in L_3$, there exists a critical point of (2.2) which is cusp.
(iii) For $(b, a) \in (B_2)$, there exist two critical points of (2.2) at $(q_{e1}, 0)$ and $(q_{e2}, 0)$ with $0<q_{e1}<q_{e2}$. $(q_{e1}, 0)$ is a saddle, $(q_{e2}, 0)$ is a center. There is a homoclinic orbit to the saddle $(q_{e1}, 0)$, in which there exists a family of periodic orbits surrounding the center $(q_{e2}, 0)$. The phase portrait is shown in Fig. 4(a).
(iv) For $(b, a) \in L_2$, $b>0$, there exist three critical points of (2.2) at $(q_{ei}, 0)$, $i=1,2,3$ with $q_{e1}<0<q_{e2}<q_{e3}$. The point $(q_{e1}, 0)$ is a cusp. $(q_{e2}, 0)$ is a saddle. $(q_{e3}, 0)$ is a center. There is a homoclinic orbit to the saddle $(q_{e2}, 0)$, in which there exists a family of periodic orbits surrounding the center $(q_{e3}, 0)$. The phase portrait is shown in Fig. 4(b).
(v) For $(b, a) \in (B_3)$, there exist four critical points of (2.2) at $(q_{ei}, 0)$, $i=1,2,3,4$ with $q_{e1}<q_{e2}<0<q_{e3}<q_{e4}$. $(q_{e1}, 0)$ and $(q_{e3}, 0)$ are saddle points. $(q_{e2}, 0)$ and $(q_{e4}, 0)$ are centers. There are two homoclinic orbits to the saddle $(q_{e1}, 0)$ and $(q_{e3}, 0)$, respectively, in which there exist two families of periodic orbits surrounding the center $(q_{e2}, 0)$ and $(q_{e4}, 0)$, respectively. The phase portrait is shown in Fig. 4(c).
(vi) For $(b, a) \in (B_4)$, there exist two critical points of (2.2) at $(q_{e1}, 0)$ and $(q_{e2}, 0)$ with $q_{e1}<0<q_{e2}$. $(q_{e1}, 0)$ is a center, $(q_{e2}, 0)$ is a saddle. There exists a family of periodic orbits surrounding the center $(q_{e1}, 0)$. The phase portrait is shown in Fig. 4(d).
(vii) For $(b, a) \in (B_5)$, there exist four critical points of (2.2) at $(q_{ei}, 0)$, $i=1,2,3,4$ with $q_{e1}<q_{e2}<q_{e3}<0<q_{e4}$. $(q_{e1}, 0)$ and $(q_{e3}, 0)$ are centers. $(q_{e2}, 0)$ and $(q_{e4}, 0)$ are saddle points. There are two homoclinic orbits to the saddle $(q_{e2}, 0)$, in which there exist two families of periodic orbits surrounding the centers $(q_{e1}, 0)$ and $(q_{e3}, 0)$; and outside which there exists a family of periodic orbits surrounding three critical points. The phase portrait is shown in Fig. 4(e).
(viii) For $(b, a) \in L_2$, $b<0$, there exist two critical points of (2.2) at $(q_{ei}, 0)$, $i=1,2$ with $q_{e1}<q_{e2}<0$. The point $(q_{e2}, 0)$ is a cusp. $(q_{e1}, 0)$ is a center. There are one homoclinic orbit to the cusp $(q_{e2}, 0)$ in which there exists a family of periodic orbits surrounding the center $(q_{e1}, 0)$. The phase portrait is shown in Fig. 4(f).

Specially, we consider that
3. The case $(2-b)c=0$ i.e., $b=2$ or $c=0$.

(1) Suppose that $b=2$, $c \neq 0$. In this case, the system (1.10) has the same topological phase portraits as the following quartic polynomial system

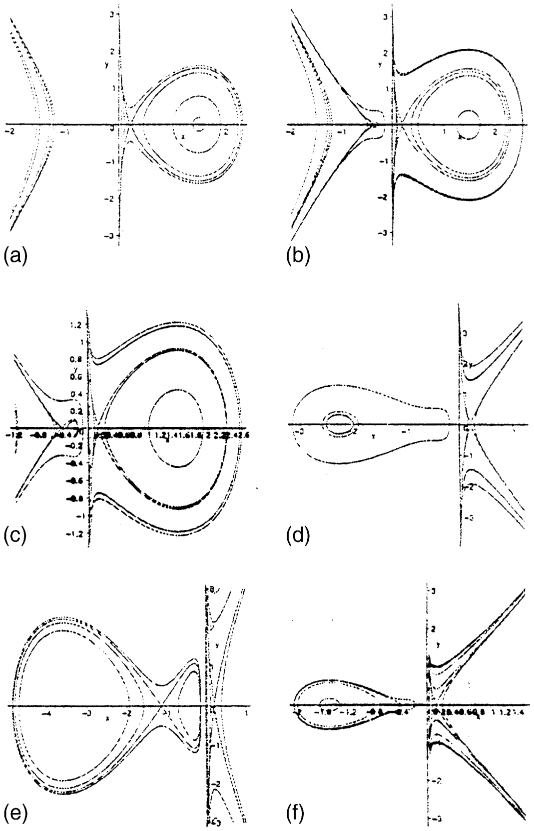

Fig. 4. Phase portraits of (2.2) when $(2-b)c<0$. (a) $(b,a)\in(B_2)$, $(c,g,b)=(-3,0.5,3)$. (b) $(b,a)\in L_2$, $b>0$, $(c,g,b)=(3,0.4519415489,3)$. (c) $(b,a)\in(B_3)$, $(c,g,b)=(-1,\sqrt{0.05},1)$. (d) $(b,a)\in(B_4)$, $(c,g,b)=(-1.5,1,-2)$. (e) $(b,a)\in(B_5)$, $(c,g,b)=(-2.4,1,-2)$. (f) $(b,a)\in L_2$, $b<0$, $(c,g,b)=(-1.5,0.6637273204,-2)$.

$$\frac{dq}{d\zeta}=q^2p,\quad \frac{dp}{d\zeta}=-\left(q^4-\frac{c^2}{4}q^2+\frac{g^2}{4}\right), \tag{2.11}$$

except for the straight lines $q=0$, where $d\xi=q^2\,d\zeta$. Clearly, the polynomial $f_1(q)=q^4-(c^2/4)q^2+g^2/4$ has 4 real zeros at $(q_{ei},0)$, $i=1$–4, if and only if $c^4-16g^2<0$, i.e., $a=(g^2/c^4)<1/16$, where $q_{e1}=-((1/8)(c^2+\sqrt{c^4-16g^2})^{1/2}$, $q_{e2}=-((1/8)(c^2-\sqrt{c^4-16g^2})^{1/2}$, $q_{e3}=((1/8)(c^2-\sqrt{c^4-16g^2})^{1/2}$, $q_{e4}=((1/8)(c^2+\sqrt{c^4-16g^2})^{1/2}$. Under this parameter condition, there are two homoclinic orbits to the saddle $(q_{e1},0)$ and $(q_{e3},0)$, respectively, in which there exist two families of periodic orbits surrounding the center $(q_{e2},0)$ and $(q_{e4},0)$, respectively. The phase portrait likes Fig. 2(c).

(2) Suppose that $c=0$. In this case, the system (1.10) has the same topological phase portraits as the following quartic polynomial system

$$\frac{dq}{d\zeta}=q^2p,\quad \frac{dp}{d\zeta}=-\left(\frac{b}{2}q^4+\frac{g^2}{4}\right), \tag{2.12}$$

except for the straight lines $q=0$, where $d\xi=q^2\,d\zeta$. Obviously, when $b<0$, (2.12) has two critical points at $(q_{ei},0)$, $i=1,2$, where $q_{e1}=-(g^2/2b)^{1/4}$, $q_{e2}=(g^2/2b)^{1/4}$. $(q_{e1},0)$ is a center, $(q_{e2},0)$ is a saddle. There exists a family of periodic orbits surrounding the center $(q_{e1},0)$. The phase portrait likes Fig. 4(d).

3. Smooth solitary and kink waves and periodic traveling wave solutions of (1.10)

In this section, we consider the existence of smooth solitary and kink waves and periodic traveling wave solutions of (1.10). We first notice that the system (2.1) has the same orbits as the system (2.2), except $q=0$. The transformation of

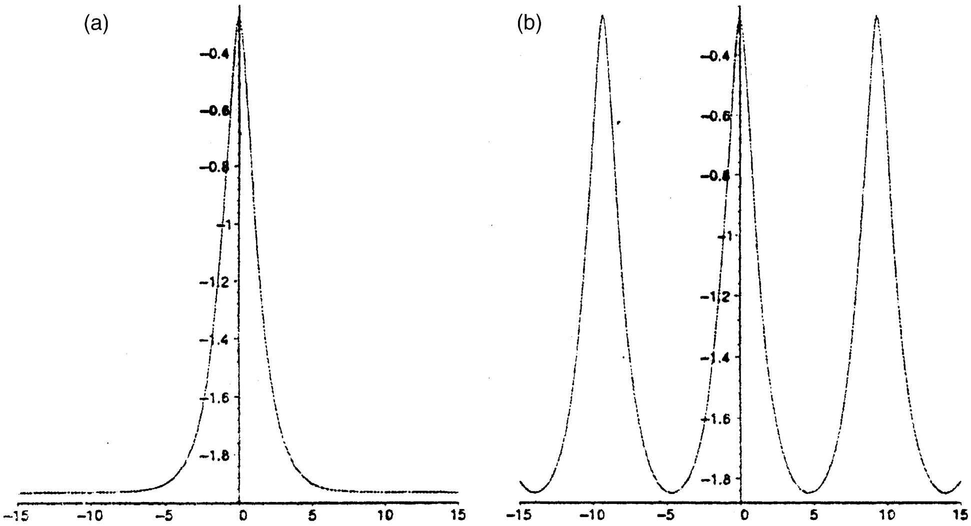

Fig. 5. Smooth solitary wave and periodic wave solutions in the case $(2-b)c>0$. (a) $q(0)=-0.2675104355, p(0)=0, (b,a)\in(A_2)$. (b) $q(0)=-0.2685104355, p(0)=0, (b,a)\in(A_2)$.

variables $d\xi=(2/|(2-b)c|)q^2\,d\zeta$ only derives the difference of the parametric representations of orbits of the systems (2.1) and (2.2) when $q\neq 0$. If an orbit of (2.2) has no intersection point with the straight lines $q=0$, then, p' is well defined in (2.1). It follows that on the (q,p)-plane the profile defined by this orbit is smooth.

Denote that $h_i=H(q_{ei},0)$ defined by (2.3). By using the discussion in Section 2, we have the following conclusion.

Theorem 3.1. *Suppose that* $(2-b)c>0$.

(i) *When* $(b,a)\in(A_2)$ *and* $L_2(b>0)$, *corresponding to* $h=h_1$, *Eq.* (1.10) *has a smooth solitary traveling wave solution of peak form. For* $h\in(h_1,h_2)$ *in* (2.3), *there are uncountably infinite many smooth periodic traveling wave solutions of* (1.10) (*see Fig.* 5).

(ii) *When* $(b,a)\in(A_3)$, *corresponding to* $h=h_1$ *and* $h=h_3$, *Eq.* (1.10) *has two smooth solitary traveling wave solutions of peak form. For* $h\in(h_2,h_1)$ *and* (h_4,h_3) *in* (2.3), *there are two families of uncountably infinite many smooth periodic traveling wave solutions.*

(iii) *When* $(b,a)\in(A_5)$, *corresponding to* $h=h_2$ *in* (2.3), *Eq.* (1.10) *has a smooth solitary traveling wave solution of peak form. For* $h\in(h_3,h_2)$ *in* (2.3), *there are uncountably infinite many smooth periodic traveling wave solutions.*

(iv) *When* $(b,a)\in(A_6)$, *corresponding to* $h=h_4$ *in* (2.3), *Eq.* (1.10) *has a smooth solitary traveling wave solution of valley form. For* $h\in(h_3,h_4)$ *in* (2.3), *there are uncountably infinite many smooth periodic traveling wave solutions.*

(v) *When* $(b,a)\in L_1$, *corresponding to* $h=h_2=h_4$ *in* (2.3), *Eq.* (1.10) *has a smooth kink solution and an unti-kink traveling wave solution. For* $h\in(h_3,h_2)$ *in* (2.3), *there are uncountably infinite many smooth periodic traveling wave solutions.*

Theorem 3.2. *Suppose that* $(2-b)c<0$.

(i) *When* $(b,a)\in(B_2)$, *corresponding to* $h=h_1$, *Eq.* (1.10) *has a smooth solitary traveling wave solution of peak form. For* $h\in(h_1,h_2)$ *in* (2.3), *there are uncountably infinite many smooth periodic traveling wave solutions of* (1.10).

(ii) *When* $(b,a)\in L_2(b>0)$, *corresponding to* $h=h_2$, *Eq.* (1.10) *has a smooth solitary traveling wave solution of peak form. For* $h\in(h_2,h_3)$ *in* (2.3), *there is a family of uncountably infinite many smooth periodic traveling wave solutions.*

(iii) *When* $(b,a)\in(B_3)$, *corresponding to* $h=h_1$ *and* $h=h_3$, *Eq.* (1.10) *has two smooth solitary traveling wave solutions of peak form. For* $h\in(h_2,h_1)$ *and* (h_4,h_3) *in* (2.3), *there are two families of uncountably infinite many smooth periodic traveling wave solutions.*

(iv) *When* $(b,a)\in L_2(b<0)$, *corresponding to* $h=h_2$ *in* (2.3), *Eq.* (1.10) *has a smooth solitary traveling wave solution of valley form. For* $h\in(h_1,h_2)$ *in* (2.3), *there is a family of uncountably infinite many smooth periodic traveling wave solutions.*

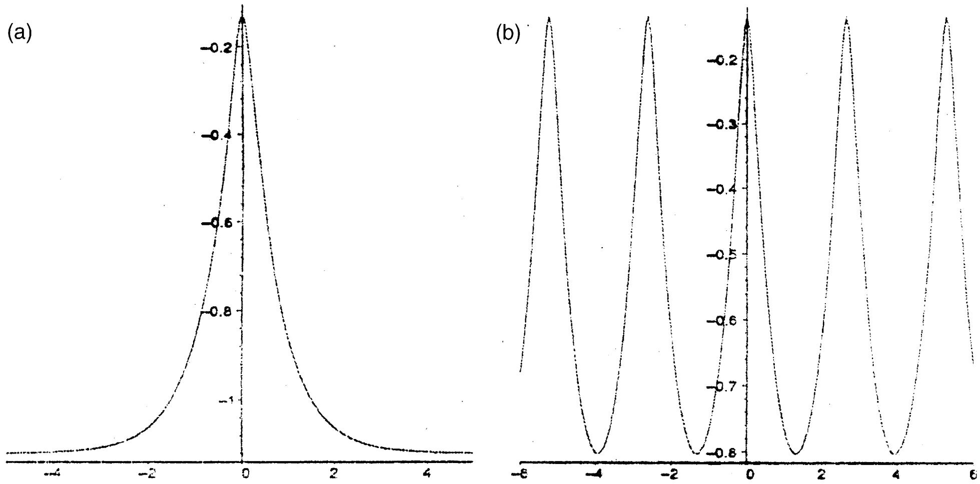

Fig. 6. Smooth solitary wave and periodic wave solutions in the case $(2-b)c<0$. (a) $(b,a)\in(B_5), q(0)=-0.1236956877, p(0)=0$. (b) $(b,a)\in(B_5), q(0)=-0.1336956877, p(0)=0$.

(v) *When* $(b,a)\in(B_5)$, *corresponding to* $h=h_2$ *in* (2.3), *Eq.* (1.10) *has two smooth solitary traveling wave solutions. One is the peak form. Another is the valley form. For* $h\in(h_1,h_2)$ *and* (h_3,h_2) *in* (2.3), *there are two families of uncountably infinite many smooth periodic traveling wave solutions* (*see Fig.* 6).

For the case $b=2, c\neq 0$, we have the same result as Theorem 3.1 (ii).

4. The existence of non-smooth periodic traveling wave solutions

In this section, we shall point out that the existence of the straight lines $\psi=0$ in the (ψ,y)-phase plane of the system (2.1) is the original reason for the appearance of non-smooth traveling wave solutions in some traveling wave models.

We now consider the case $b<0$ in Section 2. We see from Fig. 2 and Fig. 4 that when $(b,a)\in(A_4)$, L_2, (A_5), L_1, (A_6), (B_4) of Fig. 1 and Fig. 3 for $h<h_1=H(q_{e1},0)$, as h decreases and approaches $-\infty$, a segment of the right arcs of the orbits of periodic family $\{\Gamma_1^h\}$ surrounding the center $(q_{e1},0)$ will accumulate into a segment on the straight line $q=0$. We go back to (ψ,y)-plane to consider the system (2.1). Let $\epsilon=\psi$, where $\epsilon\ll 1$. Then, in a small left neighbourhood of the straight line $\psi=0$, the system (2.1) has the following relaxation oscillation form:

$$\frac{d\psi}{d\xi}=y,\quad \epsilon^2\frac{dy}{d\xi}=-\left(\frac{g^2}{4}+O(\epsilon^2)\right). \tag{4.1}$$

It is similar to the proof of Lemma 4.1 in Ref. [8]. By using (4.1), we have

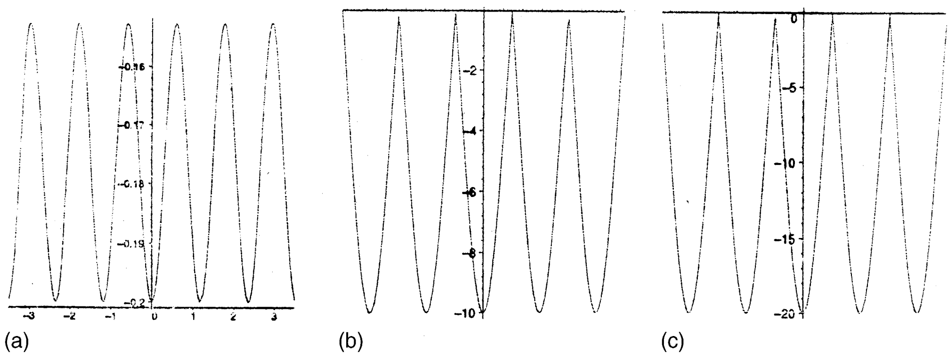

Fig. 7. From smooth periodic traveling wave evolves to periodic cusp traveling wave of (1.10) as h varies. (a) $\psi(0)=-0.2, y(0)=0$. (b) $\psi(0)=-10, y(0)=0$. (c) $\psi(0)=-20, y(0)=0$.

Lemma 4.1. *Let $(\psi, y = \psi'_\xi)$ be a point on a periodic orbit Γ_1^h of* (2.1) *surrounding $(\psi_{e1}, 0)$ with $|h|$ is sufficiently large. Then near a segment of $\psi = 0$, in a very short time interval of ξ, $y = \psi'_\xi$ rapidly jump up.*

This Lemma tell us that if a phase point (or state) (ψ, y) along the above periodic orbit Γ_1^h to move, then following ψ'_ξ rapidly jump up (changes rapidly its sign from "+" to "−" (or inverse)), ψ rapidly changes its motion direction to form a profile of cusp wave.

Similarly, when $(b, a) \in L_2, (B_5)$ of Fig. 3, *corresponding to the global family $\{\Gamma_2^h\}(h \in (-\infty, h_2)$ of periodic orbits surrounding more than one critical points of* (2.2), *the same conclusion as Lemma* 4.1 *holds.*

Thus, we have the following conclusion.

Theorem 4.2. *Suppose that the parameter group (b, a) of* (1.10) *satisfies the condition $(b, a) \in (A_4), L_2, (A_5), L_1, (A_6), (B_4), (B_5)$ of Fig.* 1 *and Fig.* 3 *Then, corresponding to the periodic family $\{\Gamma_1^h\}$ and $\{\Gamma_2^h\}$ of* (2.2), *for $h \in (-\infty, h_1)$ and $h \in (-\infty, h_2)$, Eq.* (1.10) *has uncountably infinite many periodic traveling wave solutions; when h varies from h_i ($i = 1$ or 2) to $-\infty$, these periodic traveling wave will gradually lose their smoothness, and evolve from smooth periodic traveling wave to periodic cusp traveling wave (see Fig.* 7).

For the case $c = 0$, corresponding to the periodic family $\{\Gamma_1^h\}$, the same conclusion as Theorem 4.2 *for $h \in (h_1, -\infty)$ holds.*

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