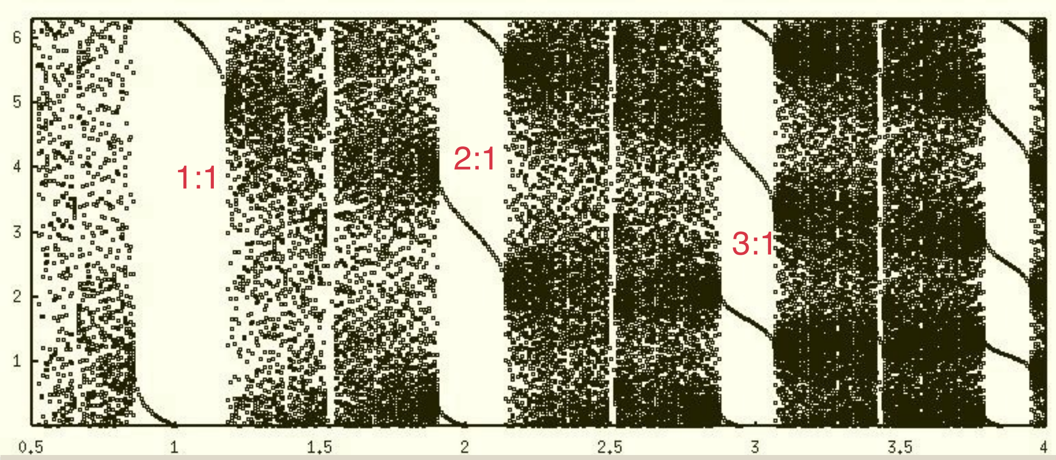
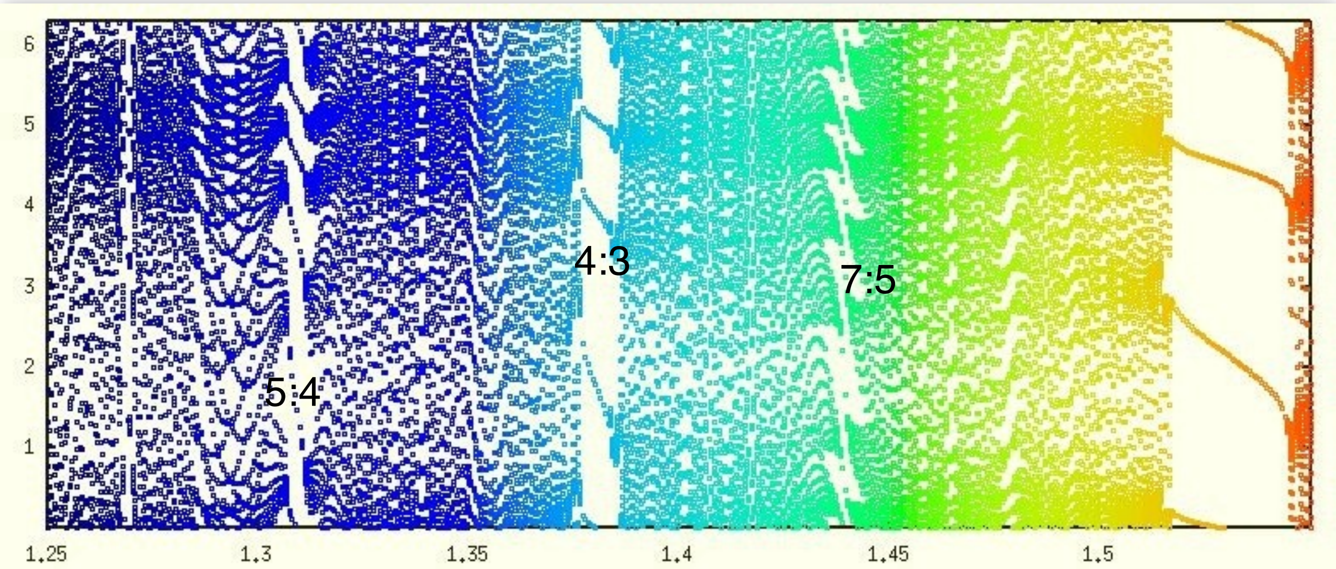
In the previous note, we developed a map for pulse coupled oscillators that were identical and had instantaneous coupling. If the frequencies are very different so that there is not 1:1 locking, it is much harder to develop a map. Instead, I just solve the ODE incrementing the appropriate variables each time one of the others crosses 2 pi. I then plot the value of the slow oscillator each time the fast oscillator "fires".



locking vs frequency of A

Note the various n:m locking regions. Between the large m:1 regions (oscillator A fires m times and B once), there are other more complex



Blow up of above

regimes. These are shown more clearly in the blowup depicted above. The 3:2 is seen at the far right as well

Now, we will consider smooth coupling using the PRC. There is no simple rigorous way to justify the model, so we will consider it as an ad hoc equation.

Let's first look at the general case of an equation on the torus:

$$x' = F(x,y)$$
$$y' = G(x,y)$$

(Note that we are now dealing with ODEs so the primes are derivatives...) We assume that F(x1+2 pi,x2)=F(x1,x2+2 pi)=F(x1,x2) and similarly for G so that the system is 2 pi-periodic in its arguments. If F,G are strictly positive, then we can also define a function h(x,y)=F/G and get

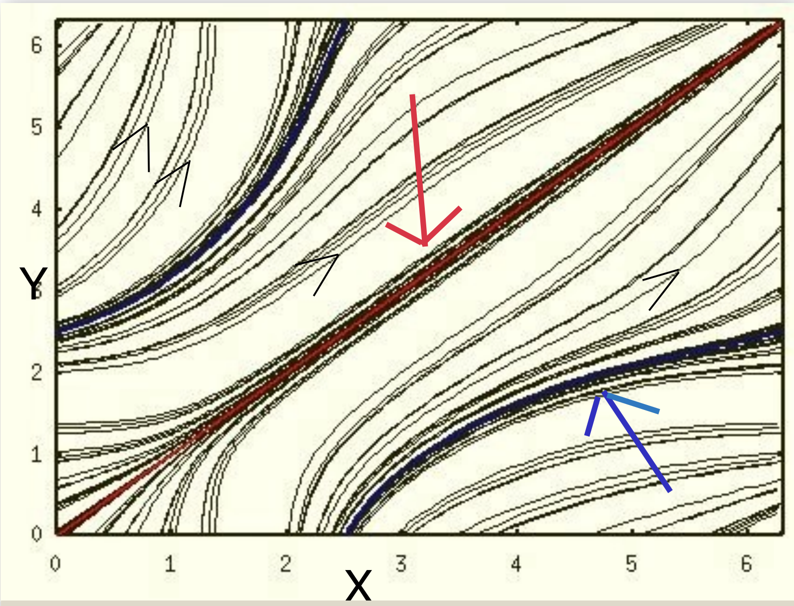
$$dx/dy = h(x,y)$$

If h(x,y) is continuous, then there is a solution x=P(y;x0) which exists for all time and satisfies P(y;x0+2 pi)=P(y;x0).
Consider M(x0)=P(2 pi;x0). This defines a nice map on the circle x0 -> M(x0), and so we can define the rotation number, rho, for this map. We can think of rho as the number of times that x goes around the circle compared to y.

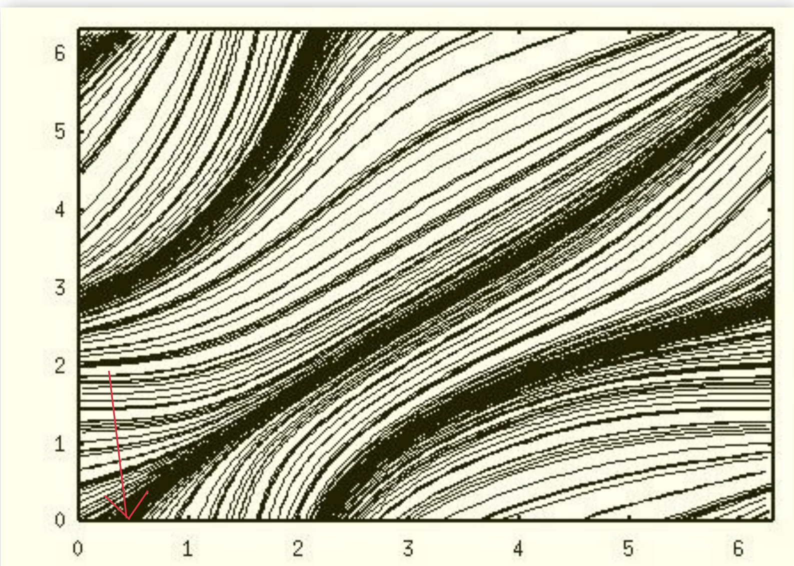
We can create equations on the torus using "pulse" coupling and the PRC. Let D(x) be the PRC for an oscillator. Consider

$$x' = w1 + P(y)\, D(x) \quad (*)$$
$$y' = w2 + P(x)\, D(y)$$

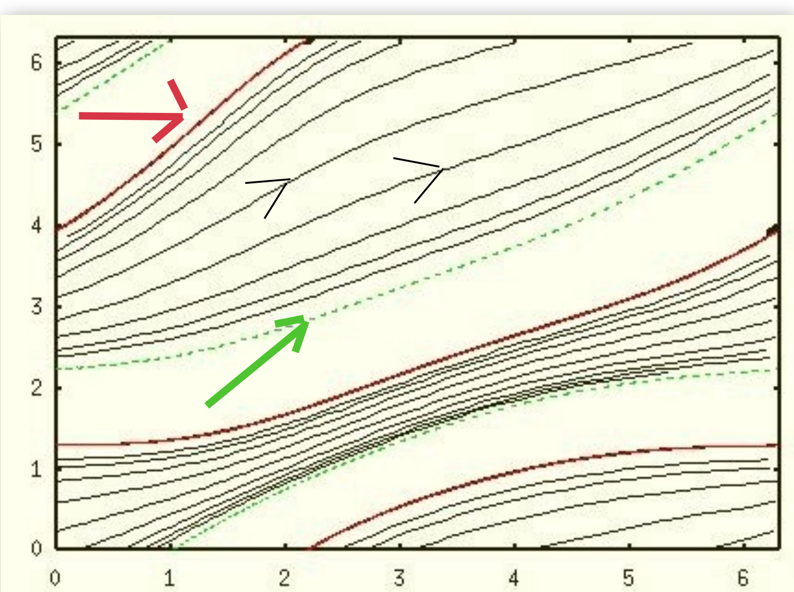
In absence of coupling P=0 and these guys flow around with a rotation number rho=w1/w2. If this is irrational, they densely cover the torus. If it is rational, it generates a periodic solution. If P(x) is the Dirac delta function, then this set of equations reduces to our map-based ones. For, in this case P=0 except when y (or x) hits 2 pi. When P is zero and w1=w2=1, then y is just y0 + t. If x hits 2 pi at t=0, then y -> y+D(y)=PTC(y), and so forth! So, think of (*) as the generalization of our map. It is also much easier to see how to make big networks in this case. Suppose w2=1, P(x)=(1+cos(x)), and D(x)=-a sin(x). Thus, D(x) is our usual PRC and P(x) has a peak when x=0 and is symmetric about x=0. It is a big fat pulse!



identical freq



w1=1.2



2:1 locking

The picture on the left shows the dynamics of (*) for a=0.5. Note that trajectories are attracted to the diagonal (red arrow) which is the stable synchronous solution and that there is also an unstable (blue arrow) "anti-phase" solution where the two are a half cycle out of phase. Both of these solutions represent solutions with rotation number 1. If I change the frequency of x by a little, the rotation number stays the same, but the stable solution is no longer along the diagonal, is is offset slightly. You can see that when Y=0, X is a bit greater than 0 (red arrow) showing that X is a bit advanced from Y in keeping with the fact that it has a higher frequency. Finally, the last figure shows the dynamics when w1=2. There is a very clear 2:1 locked stable solution (green dotted curve,green arrow) and an unstable solution (red) that is also 2:1. Note how X goes through two cycles as Y goes through 1.

HW: With this same model can you show a 3:1 locked solution? How about a 3:2? (Note that I am not saying these exist - they might not. In fact, you may have to put more Fourier coefficients in the P(x)! Try simulations with P(x)=exp(-(1-cos(x)))