Aside Residue Theorem

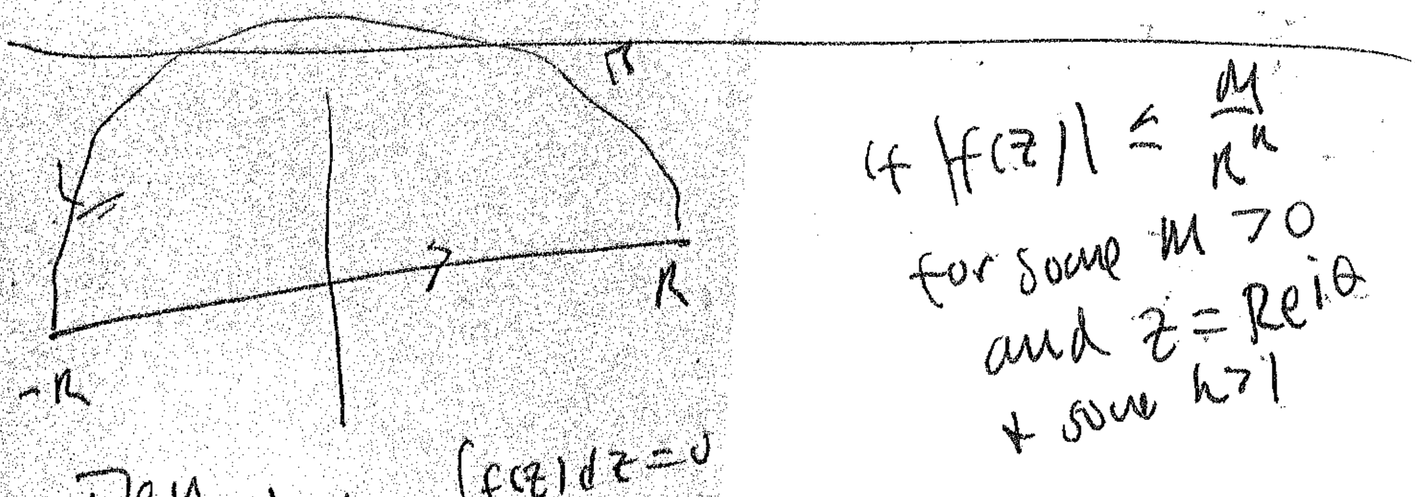
if $f(z)$ is analytic inside & on a simple closed curve except at $z=a$ where it has a pole of order n:

$$f(z) = \frac{a_{-n}}{(z-a)^n} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \cdots$$

Then $\oint_C f(z)\,dz = 2\pi i\, a_{-1}$

$$a_{-1} = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{(n-1)}}{dz^{n-1}}\left((z-a)^n f(z)\right)$$

eg if $f(z) = \frac{g(z)}{z-a}$ Then $\frac{1}{2\pi i}\oint f(z)\,dz = g(a)$



if $|f(z)| \le \frac{M}{R^n}$

for some $M > 0$

and $z = Re^{i\theta}$

& some $n > 1$

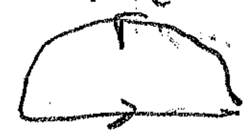
Then $\lim_{R \to \infty} \int_\Gamma f(z)\,dz = 0$

so can use this to evaluate integrals

CR

$\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$ use same contour

$z^4+1=0$ when $z = e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}} e^{\frac{5\pi i}{4}} e^{\frac{7\pi i}{4}}$

only poles at $e^{\frac{i\pi}{4}}$ $e^{\frac{3\pi i}{4}}$ lie in C 

use L'Hopital to get

$$\lim_{z \to e^{\frac{\pi i}{4}}} (z-e^{\frac{\pi i}{4}}) \frac{1}{z^4+1} = \frac{1}{4} e^{-3\pi i/4}$$

$$\lim_{z \to e^{\frac{3\pi i}{4}}} (z-e^{\frac{3\pi i}{4}}) \frac{1}{z^4+1} = \frac{1}{4} e^{-\frac{9\pi i}{4}}$$

$$\oint_C \frac{dz}{z^4+1} = 2\pi i\left(\frac{1}{4} e^{-\frac{3\pi i}{4}} + \frac{1}{4} e^{-\frac{9\pi i}{4}}\right) = \frac{\pi\sqrt{2}}{2}$$

$$\int_{-R}^{R} \frac{dz}{z^4+1} + \underbrace{\int_{\Gamma} \frac{dz}{z^4+1}}_{\to 0} = \frac{\pi\sqrt{2}}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{2} \;!$$

Kuramoto with just sine coupling ①

$$\frac{d\theta_j}{dt} = \omega_j + \frac{K}{N}\sum_{k=1}^{N} \sin(\theta_k - \theta_j) \qquad \omega_j \in g(\omega) \text{ density}$$

$$\frac{1}{N}\sum_k \sin(\theta_k - \theta_j) = \mathrm{Im}\left\{e^{-i\theta_j}\frac{1}{N}\sum_{k=1} e^{i\theta_j}\right\}$$

$$= \mathrm{Im}\left[r e^{-i\theta_j}\right]$$

$r = \frac{1}{N}\sum_k e^{i\theta_k}$ Let $N \to \infty$, + Let

$f(\omega, \theta, t) =$ prob den ~~that~~ θ & ω, t

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}\left(\left[\omega + \frac{K}{2i}\left[r e^{-i\theta} - \bar{r} e^{i\theta}\right]f\right)\right) = 0$$

since $\mathrm{Im}\, r e^{-i\theta} = \dfrac{r e^{-i\theta} - \bar{r} e^{i\theta}}{2i}$

$\left(\dfrac{a+bi - a-bi}{2i} = b\right) = \mathrm{Im}\, r e^{-i\theta}$

$$\boxed{r = \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} d\omega\, f(\omega, \theta, t) e^{i\theta}}$$

We will show that the dynamics of this PDE actually lies in very low dimension

Expand f in Fourier series!

$$f = \frac{g(\omega)}{2\pi}\left\{1 + \left[\sum_{n=1}^{\infty} f_n(\omega,t)\, e^{in\theta} + c.c.\right]\right\}$$

Let us suppose $f_n(\omega,t) = \alpha(\omega,t)^n$

with $|\alpha|<1$. Lets see if this actually

solves the equation

Clearly we can factor the

$\frac{g(\omega)}{2\pi}$ out of this so lets just

look at $f = \frac{g(\omega)}{2\pi} F$

$$F = 1 + \sum_{n=1}^{\infty} \alpha^n e^{in\theta} + \sum_{n=1}^{\infty} \bar{\alpha}^n e^{-in\theta}$$

$$\frac{\partial F}{\partial t} = \sum n\alpha^{n-1}\frac{d\alpha}{dt} e^{in\theta} + \sum n\bar{\alpha}^{n-1} e^{-in\theta}\frac{d\bar{\alpha}}{dt}$$

$$\left[\frac{k}{2i}\left(r e^{-i\theta} - \bar{r} e^{i\theta}\right) + \omega\right] F =$$

$$\omega\left[1 + \sum_{n=1}^{\infty} \alpha^n e^{in\theta} + c.c\right]$$

$$+ \frac{k}{2i} r\left[e^{-i\theta} + \sum_{n=1}^{\infty} \alpha^n e^{i(n-1)\theta} + \bar{\alpha}^n e^{-i(n+1)\theta}\right]$$

$$- e^{i\theta} \quad \alpha^n e^{i(n+1)\theta} + \bar{\alpha}^n e^{-i(n-1)\theta}\Big]$$

3) Diff wrt θ:

$$\omega \sum_{n=1}^{\infty} \alpha^n i n e^{in\theta} - \bar{\alpha}^n i n e^{-in\theta}$$

$$+ \frac{kr}{2i}\left[-ie^{-i\theta} + \sum_{n=1}^{\infty} \alpha^n i(n-1) e^{i(n-1)\theta} - \bar{\alpha}^n i(n+1) e^{-i(n+1)\theta}\right.,$$

$$- \frac{k\bar{r}}{2i}\left[ie^{i\theta} + \sum_{n=1}^{\infty} \alpha^n i(n+1) e^{i(n+1)\theta} - \bar{\alpha}^n i(n-1) e^{-i(n-1)\theta}\right]$$

$e^{i\theta}$ term:

$$1\alpha^0 \frac{d\alpha}{dt} + i\omega\alpha + \frac{kr}{2}\alpha^2 - \frac{k\bar{r}}{2} = 0$$

$e^{2i\theta}$

$$2\alpha \frac{d\alpha}{dt} + 2i\alpha^2\omega + \frac{2\alpha^3 kr}{2} - \frac{k\bar{r}2\alpha}{2} = 0$$

$\alpha \neq 0 \Rightarrow$

$$\frac{d\alpha}{dt} + i\alpha\omega + \frac{kr}{2}\alpha^2 - \frac{k\bar{r}}{2} = 0$$

and so on!!

#4 so we have

$$\frac{d\alpha}{dt} + \frac{K}{2}[r\alpha^2 - \bar{r}] + i\omega\alpha = 0$$

~~what go~~ with

$$r = \int_0^{\infty} d\theta \int_{-\infty}^{\infty} d\omega\, f e^{i\theta}$$

$$\bar{r} = \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} d\omega\, f\, e^{-i\theta} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} g(\omega) e^{-i\theta} F\, d\omega$$

$$= \int_{-\infty}^{\infty} g(\omega)\, \alpha(\omega, t)\, d\omega$$ since only coeff $e^{i\theta}$ of F matters

$f(\omega, \theta, t)$ can be summed un

$$= \frac{g(\omega)}{2\pi}\left[\frac{(1-|\alpha|)(1+|\alpha|)}{(1-|\alpha|^2) + 4|\alpha|\sin^2(\frac{1}{2}(\theta-\psi))}\right]$$

Where $\alpha = |\alpha| e^{-i\psi}$

As $\alpha \to 1 \to \delta(\theta - \psi)\frac{g(\omega)}{2\pi}$ as expected (perfect sync)

#5 ODE + integral equation doesn't still inf dim since need for each ω.
(could approx for some finite dist)

Lets simplify more.

Let $g(\omega) = \frac{\Delta}{(\omega-\omega_0)^2+\Delta^2}\frac{1}{\pi}$

By rescaling we can assume $\omega_0 = 0$
+ if we $\frac{d\theta_j}{dt} = \omega_j + \frac{K}{N}\Sigma(\ )$

Divide by Δ + rescale t

$$\frac{d\theta_j}{dt} = \frac{\omega_j}{\Delta} + \frac{K/\Delta}{N}\Sigma(\ )$$

so replace with new ω_j/Δ + K/Δ

$g(\omega) = \frac{1}{\omega^2+1}\frac{1}{\pi}$ Lorentzian dist

$$\frac{1}{\pi}\frac{1}{\omega^2+1} = \frac{1}{2\pi i}\left[\frac{1}{\omega-i} - \frac{1}{\omega+i}\right]$$ Poles at $\pm i = \omega$

Take $\omega = -i$

$$\frac{d\alpha}{dt} = \frac{K}{2}\left[\bar{r} - r\alpha^2\right] - i\omega\alpha$$

If ~~r=0, eq~~ ~~... for~~ $K=0$ $\frac{d\alpha}{dt} = -i\omega\alpha$ else $\frac{d\alpha}{dt} = \alpha$

α will grow unstable

#6 Assume that $\alpha(\omega, t)$ can be extended to complex ω & apply residue theorem:

$$\bar{r} = \int_{-\infty}^{\infty} d\omega \, \alpha(\omega, t) \, g(\omega)$$

single pole at $\omega = -i$

$$= \alpha(-i, t)$$

$$\bar{r} = \bar{\alpha}(-i, t)$$

Write $r = \rho e^{-i\phi}$

$$\boxed{\frac{d\alpha(-i,t)}{dt} + \frac{k}{2}\left(r\alpha^2(-i,t) - \bar{r}\right) + \alpha(-i,t) = 0}$$

$$\cancel{\frac{dr}{dt} = \left(\frac{d\rho}{dt} - i\rho\dot{\phi}\right)e^{-i\phi}}$$

$$\frac{d\alpha(-i,t)}{dt} + \frac{k}{2}\left(\bar{\alpha}\alpha^2 - \alpha\right) + \alpha = 0$$

write $\alpha = \rho e^{i\psi}$

$$\frac{d\rho}{dt} + \frac{k}{2}\left(\rho^3 - \rho\right) + \rho = 0$$

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$$\frac{d\rho}{d\tau} = \rho\left[\frac{k}{2} - 1\right] - \frac{k}{2}\rho^3$$

$\rho = 0$ is stable as long as $k < 2$ when

$k > 2$ $\rho \to \sqrt{1 - \frac{2}{k}}$

Recall $g(0) = \frac{1}{\pi}$ $k_c = \frac{2}{g(0)\pi}$ from

before!