HW #1

1. Solve $\dot{x} = x(r - kx)$, assuming $x(0) \in (0, \frac{r}{k})$, where $r, k > 0$

2. Write down equations + analyze the following model:

$X \xrightarrow{r} *$, $2X \xrightarrow{k} 3X$, $3X \xrightarrow{\beta} 2X$ ("three's a crowd")

In particular, assume all constants are positive + find all the equilibria + their stability. If there are different qualitative pictures, sketch the different phase lines

3. Consider the general LV system $\boxed{\dot{x}_i = x_i(r_i + \sum_{j=1}^{N} a_{ij} x_j) \quad i = 1, .., N}$ EQ 1

Suppose $x_i(0) > 0$ $\forall i$. Prove $x_i(t) > 0$ for all finite time. (Hint: Suppose that $x_i(t^*) = 0$ for some t^*. Show that this violates the uniqueness theorem of ODES)

(4) Suppose Eq 1 has a periodic orbit $x_i(t) = u_i(t)$, with period T + $u_i(t) > 0$.

Let $\bar{u}_i = \frac{1}{T}\int_0^T u_i(t)dt$ be the average over one cycle.

<u>Prove</u> $r_i + \sum_{j=1}^{N} a_{ij}\bar{u}_j = 0$, $i = 1, .., N$ + thus that there is an equilibrium point in the interior of the positive orthant

5. Put the competition equations into the form, $\dot{x}_1 = r_1 x_1(1 - \frac{x_1}{K_1} - \alpha_{12}\frac{x_2}{K_2})$

$\dot{x}_2 = r_2 x_2(1 - \frac{x_2}{K_2} - \alpha_{21}\frac{x_1}{K_1})$ eg. express r_j, K_j, α_{jk} in terms of a-f

6. In the proof of the nonexistence of isolated periodic orbits, show that $\alpha a + \beta d$ is the trace of the Jacobian evaluated at the interior fixed point.

7. Use #6 to show that the general 2D LV system has a continuum of periodic orbits iff the eigenvalues of the interior equilibrium are pure imaginary eg: Trace $= 0$, det > 0

8. In general, how many equilibrium points are there in the general LV model: $\dot{x}_i = x_i(r_i + \sum_{j=1}^{N} a_{ij}x_j)$, $j=1,..,N$
(They need not be positive, I am just counting the total)

9. In the general 2D LV model:
$\dot{x} = x(a+bx+cy)$, $\dot{y} = y(d+ex+fy)$
determine stability of the three points: $(0,0)$, $(x^*,0)$, $(0,y^*)$
Suppose there is no stable interior equilibrium + all other equilibria are unstable. What happens? (Hint use P.B.T.).

10. Consider the 3 variable LV model:
$\dot{x}_i = x_i\left(\sum_{j=1}^{3} a_{ij}(1-x_j)\right)$, where $a_{11}=a_{22}=0.1$, $a_{12}=a_{23}=-1$
$a_{31}=\gamma$, $a_{32}=2$, $a_{33}=\frac{1}{2}$, $a_{13}=a_{21}=0$
Clearly $\bar{x}_1=\bar{x}_2=\bar{x}_3$ is the interior equilibrium
find conditions on $\gamma>0$ st $(1,1,1)$ is A.S. Show by simulation that this model admits a stable periodic limit cycle solution for some γ. I will put the XPP code online, but you can use any software.