The dipole distribution is frequently represented as \( \Delta = -\delta' \) since \( \delta'(0) = \lim_{\epsilon \to 0} \frac{\delta(x)-\delta(x-\epsilon)}{2\epsilon} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} (\delta(x) - \langle \delta_{\epsilon}, \phi \rangle) \) but since \( \delta \) is not a function it is certainly not differentiable in the usual sense. We shall define the derivative of \( \delta \) momentarily.

One often sees the notation \( \int_{-\infty}^{\infty} \delta(x-t)\phi(t)dt = \phi(x) \). It should always be kept in mind that this is simply a notational device to represent the distribution \( \langle \delta_{x}, \phi \rangle \) and is in no way meant to represent an actual integral or that \( \delta(x-t) \) is an actual function. The notation \( \delta(x-t) \) is a "SYMBOLIC FUNCTION" for the delta distribution.

The correct way to view \( \delta_{x} \) is as an operator on the set of test functions. We should never refer to pointwise values of \( \delta_{x} \) since it is not a function, but an operator on functions. The operation \( \langle \delta_{x}, \phi \rangle = \phi(0) \) makes perfectly good sense and we have violated no rules of integration or function theory to make this definition.

The fact that some operators can be viewed as being generated by functions through normal integration should not confuse the issue. \( \delta_{x} \) is not such an operator. Another operator that is operator valued but not pointwise valued is the operator (not a linear functional) \( L = d/dx \). We know that \( d/dx \) cannot be evaluated at the point \( x = 3 \) for example, but \( d/dx \) can be evaluated pointwise only after it has first acted on a differentiable function \( u(x) \). Thus, \( du/dx = u'(x) \) can be evaluated at \( x = 3 \), only after the operand \( u(x) \) is known. Similarly, \( \langle \delta_{x}, \phi \rangle \) can be evaluated only after \( \phi \) is known.

Although distributions are not always representable as integrals, their properties are nonetheless always defined to be consistent with the corresponding property of inner products. The following are some properties of distributions that result from this association.

1. If \( t \) is a distribution and \( f \in C^\infty \) then \( ft \) is a distribution whose action is defined by \( \langle ft, \phi \rangle = \langle t, f\phi \rangle \). For example, if \( f \) is continuous \( f(x)\delta = f(0)\delta \).

2. If \( f \) is continuously differentiable at \( 0 \), \( f'\delta = -f'(0)\delta + f(0)\delta' \). This follows since

\[
\langle f\delta', \phi \rangle = \langle \delta', f\phi \rangle = -\langle f\phi' \rangle_{x=0} = -\langle f'(0)\delta + f(0)\delta' \rangle_{x=0} = -\langle f'(0)\phi(0) - f(0)\phi'(0) \rangle.
\]

3. Two distributions \( t_{1} \) and \( t_{2} \) are said to be equal on the interval \( a < x < b \) if for all test functions \( \phi \) with support in \( [a, b] \), \( \langle t_{1}, \phi \rangle = \langle t_{2}, \phi \rangle \). Therefore it is often said (and this is unfortunately misleading) that \( \delta(x) = 0 \) for \( x \neq 0 \).

4. The derivative \( t' \) of a test functions \( \phi \in D \) functions

\[
\langle f', \phi \rangle = \int_{-\infty}^{\infty} f'(t)\phi(t)dt.
\]

Since \( \phi(x) \) has compact support, \( f(t) \) can be compactly supported at \( x = \pm \infty \).

If \( t \) is a distribution, then in \( D \), \( \{\phi_{n}\} \) is also a zero distribution since \( \langle \phi_{n}, \phi \rangle = \langle \phi_{n}' \phi \rangle = \langle \phi_{n}(\phi') \rangle = \langle (\phi'^{(n)}) \phi \rangle \). It follows that for any distribution \( t \), if it exists and its action is

\[
\langle t^{(n)}, \phi \rangle = \langle (t')^{(n)}, \phi \rangle = \langle \phi''^{(n)} \rangle,
\]

Thus any \( L^2 \) function has distributional derivative.

Examples

1. The Heaviside distribution

\[
\langle H', \phi \rangle = \langle H, \phi' \rangle
\]

since \( \phi \) has compact support.

2. The derivative of the delta function is the negative of the delta function

\[
\langle \delta'^{(n)}, \phi \rangle = \langle \delta^{(n)} \phi \rangle
\]

which is the negative of \( \delta^{(n)} \).

3. Suppose \( f \) is continuously differentiable at which \( f \) has jump discontinuity, its distribution has derivative

\[
\langle f', \phi \rangle = \langle f, \phi' \rangle = \langle (f' - f) \phi \rangle.
\]
4.1 DISTRIBUTIONS AND THE DELTA FUNCTION

\[ \langle t(x - \xi), \phi \rangle = \langle t, \phi(x + \xi) \rangle, \]
even though pointwise values of \( t \) may not have meaning. It follows for example that \( \delta(x - \xi) = \delta_\xi \) and \( \delta(ax) = \delta(x)/|a| \).

4. The derivative \( t' \) of a distribution \( t \) is defined by \( \langle t', \phi \rangle = -\langle t, \phi' \rangle \) for all test functions \( \phi \in D \). This definition is natural since, for differentiable functions

\[ \langle f', \phi \rangle = \int_{-\infty}^{\infty} f'(x)\phi(x)\,dx = -\int_{-\infty}^{\infty} f(x)\phi'(x)\,dx = -\langle f, \phi' \rangle. \]

Since \( \phi(x) \) has compact support, the integration by parts has no boundary contributions at \( x = \pm \infty \).

If \( t \) is a distribution, then \( t' \) is also a distribution. If \( \{\phi_n\} \) is a zero sequence in \( D \), then \( \{\phi_n'\} \) is also a zero sequence, so that

\[ \langle t', \phi_n \rangle = -\langle t, \phi_n' \rangle \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

It follows that for any distribution \( t \), the \( n \)th distributional derivative \( t^{(n)} \) exists and its action is

\[ \langle t^{(n)}, \phi \rangle = (-1)^n \langle t, \phi^{(n)} \rangle. \]

Thus any \( L^2 \) function has distributional derivatives of all orders.

**Examples**

1. The Heaviside distribution \( \langle H, \phi \rangle = \int_0^{\infty} \phi(x)\,dx \) has derivative

\[ \langle H', \phi \rangle = -\int_0^{\infty} \phi'(x)\,dx = \phi(0) \]

since \( \phi \) has compact support, so that \( H' = \delta_0 \).

2. The derivative of the \( \delta \)-distribution is

\[ \langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0) \]

which is the negative of the dipole distribution.

3. Suppose \( f \) is continuously differentiable except at the points \( x_1, x_2, \ldots, x_n \) at which \( f \) has jump discontinuities \( \Delta f_1, \Delta f_2, \ldots, \Delta f_n \), respectively. Its distribution has derivative given by

\[ \langle f', \phi \rangle = -\langle f, \phi' \rangle = -\int_{-\infty}^{\infty} f(x)\phi'(x)\,dx \]
\[
\begin{align*}
&= \int_{-\infty}^{x_1} f(x) \phi'(x) \, dx \\
&\quad + \int_{x_1}^{x_2} f(x) \phi'(x) \, dx + \cdots + \int_{x_n}^{\infty} f(x) \phi'(x) \, dx \\
&= \int_{-\infty}^{\infty} \frac{df}{dx} \phi(x) \, dx + \sum_{k=1}^{n} \Delta f_k \phi(x_k).
\end{align*}
\]

It follows that the distributional derivative of \( f \) is
\[
f' = \frac{df}{dx} + \sum_{k=1}^{n} \Delta f_k \delta_{x_k},
\]
where \( \frac{df}{dx} \) is the usual calculus derivative of \( f \), wherever it exists.

4. For \( f(x) = |x| \), the distributional derivative of \( f \) has action
\[
(f', \phi) = -\langle f, \phi' \rangle = -\int_{-\infty}^{\infty} |x| \phi'(x) \, dx
\]
\[
= \int_{-\infty}^{0} x \phi'(x) \, dx - \int_{0}^{\infty} x \phi'(x) \, dx
\]
\[
= -\int_{-\infty}^{0} \phi(x) \, dx + \int_{0}^{\infty} \phi(x) \, dx
\]
\[
= -\int_{-\infty}^{\infty} \phi(x) \, dx + 2\int_{0}^{\infty} \phi(x) \, dx
\]
so that \( f' = -1 + 2H(x) \), and \( f'' = 2 \delta_0 \).

\section*{Definition}
A sequence of distributions \( \{t_n\} \) is said to converge to the distribution \( t \) if their actions converge in \( \mathbb{R} \), that is, if
\[
\langle t_n, \phi \rangle \to \langle t, \phi \rangle \quad \text{for all} \quad \phi \in D.
\]
This convergence is called convergence in the sense of distribution or \textit{weak convergence}.

If the sequence of distributions \( t_n \) converges to \( t \) then the sequence of derivatives \( t'_n \) converges to \( t' \). This follows since
\[
\langle t'_n, \phi \rangle = -\langle t_n, \phi' \rangle \to -\langle t, \phi' \rangle = \langle t', \phi \rangle
\]
for all \( \phi \) in \( D \).

\section*{Example}
The sequence \( \{t_n\} \) is a sequence of distributions (pointwise) and as a sequence it converges to the zero distribution.

Using distributions, we can give a meaning to the distributional derivative to many objects that do not have a derivative in the usual sense.

\section*{Definition}
The differential equation of distribution (i.e., \( f' = g \)) is called an \textit{integral equation}.

\section*{Examples}
1. To solve the equation \( f' = g \) for a distribution \( u \) for \( x > 0 \).
   This rather mundane equation \( f' = g \) has many solutions. Although it is somewhat non-intuitive, we want to define what we mean by \( f' = g \). Said another way, we want to define the action of \( u \) as a function.
   We start by saying that the action of \( u \) is a function. Said another way, we start by saying that the action of \( u \) is a function on \( D' \), the set of test functions. We first define \( \int_{-\infty}^{\infty} \phi \, dx = 0 \). Certain \( \int_{-\infty}^{\infty} \psi \, dx \) are not defined.
   We define \( \phi = \int_{-\infty}^{\infty} \psi \, dx \) is a function of compact support. It is defined on \( D' \).

   Now comes the trick, \( \int_{-\infty}^{\infty} \phi_0(x) \, dx = 1 \). A combination of \( \phi_0 \) and \( \phi \) is defined as
\[
\phi(x) = \phi_0(x) + \phi(x).
\]