Series solutions

1 Regular series

We will study the solutions to certain second order linear differential equations. We will start with equations that are analytic at $t = 0$ and then turn to those that have singularities. Recall that a function is *analytic* at a point $t = t_0$ if it has a convergent Taylor series at that point. We are interested first in equations of the form:

$$x'' + p(t)x' + q(t)x = 0 \tag{1}$$

where $p(t), q(t)$ have Taylor series at $t = 0$. The basic idea is to assume that there is a solution of the form:

$$x(t) = a_0 + a_1 t + a_2 t^2 + \dots$$

and then try to prove that the series converges. In general, we will get an iterative series of equations for a_j interms of already known coefficients. The first two coefficients will be determined by the initial conditions:

$$x(0) = a_0 \quad x'(0) = a_1.$$

Lets do a simple example that we already can solve:

$$x'' + x = 0$$

Substituting the series into the derivative:

$$x''(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + \dots j(j-1)t^{j-2}a_j +$$

and equating the coefficients we get

$$\begin{aligned}
0 &= a_0 + 2a_2 \\
0 &= a_1 + 6a_3 \\
0 &= a_2 + 12a_4 \\
0 &= a_{2j-1} + (2j+1)(2j)a_{2j+1} \\
0 &= a_{2j} + (2j+2)(2j+1)a_{2j+2}
\end{aligned}$$

Thus we get a recursive relationship for the coefficients:

$$\begin{aligned}
a_{2j+2} &= -\frac{1}{(2j+2)(2j+1)}a_{2j} \\
a_{2j+1} &= -\frac{1}{(2j+1)(2j)}a_{2j-1}
\end{aligned}$$

From this it is easy to see that

$$a_{2j+1} = \frac{(-1)^j}{(2j+1)!}a_1$$

and

$$a_{2j} = \frac{(-1)^j}{(2j)!}a_0$$

The series breaks into an even part that depends only on a_0 and an odd part that depends only on a_1. Writing the two series down, we immediately recognize them:

$$x_{even}(t) = a_0(1 - t^2/(2!) + t^4/(4!)) \dots = a_0 \cos t$$

and

$$x_{odd}(t) = a_1(t - t^3/(3!) + t^5/(5!) \dots) = a_1 \sin t.$$

Thus, we recover the already known solutions, $\sin(t), \cos(t)$.

Let's try an example that we don't already know:

$$x'' - tx = 0$$

Once again we expand in a series and obtain

$$\begin{aligned} 2a_2 &= 0 \\ 6a_3 - a_0 &= 0 \\ 12a_4 - a_1 &= 0 \\ 20a_5 - a_2 &= 0 \\ j(j-1)a_j - a_{j-3} &= 0 \end{aligned}$$

This implies that

$$a_j = \frac{1}{j(j-1)} a_{j-3}.$$

Note that $a_2 = a_5 = a_8 \ldots = 0$, $a_3, a_6, \ldots, a_{3j}$ depend on a_0 and $a_4, a_7, \ldots, a_{3j+1}$ depend on a_1. Note that the series converges for all t since

$$\left| \frac{a_{j+3} t^{j+3}}{a_j t^j} \right| = |t|^3 \frac{1}{j(j-1)}$$

and this tends to zero as $j \to \infty$ so by the ratio test the series converges for all t. Some rearranging of terms yields

$$\begin{aligned} a_{3j} &= \frac{1}{(3j)(3j-1)} \frac{1}{(3j-3)(3j-4)} \cdots \frac{1}{3 \cdot 2} a_0 \\ a_{3j+1} &= \frac{1}{(3j+1)(3j)} \frac{1}{(3j-2)(3j-3)} \cdots \frac{1}{4 \cdot 3} a_1. \end{aligned}$$

This equation is called *Airy's* equation and the two linearly independent solutions are called Airy functions.

In general we have the following result. Consider

$$x'' + p(t)x' + q(t)x = 0 \quad x(0) = a_0 \quad x'(0) = a_1$$

and suppose that

$$p(t) = \sum_{k=0}^{\infty} p_k t^k, \quad q(t) = \sum_{k=0}^{\infty} q_k t^k$$

converge in some interval J containing $t = 0$. Then

$$x(t) = \sum_{j=0}^{\infty} a_j t^j$$

where

$$(j+2)(j+1)a_{j+2} + \sum_{k=0}^{j} [p_{j-k} a_{k+1}(k+1) + q_{j-k} a_k] = 0$$

and the series converges on the whole interval J.

As a last application of regular series, we apply it to the Legendre's equation which arises in the study of certain partial-differential equations defined in a spherical domain:

$$(1-t^2)x'' - 2tx' + p(p+1)x = 0.$$

We rewrite this as

$$x'' - \frac{2t}{1-t^2} y' + \frac{p(p+1)}{1-t^2}.$$

We use the binomial expansion to expand the denominator

$$(1 - t^2)^{-1} = 1 + t^2 + t^4 + \ldots + t^{2j} + \ldots.$$

Note that this series converges only for $|t| < 1$. Using the above recursion we get after some manipulation (I will leave it to you to verify this) that

$$a_{j+2} = -\frac{(p + j + 1)(p - j)}{(j + 2)(j + 1)} a_j.$$

This series converges from the ratio test as long as $|t| < 1$. Furthermore note that the even terms depend only on a_0 and the odd on a_1. Note that if p is an integer, the series is finite and only goes out to $p + 2$ terms. If $p = n$ is an integer, we call the solutions the Legendre polynomials:

$$P_0 = 1,\ P_1 = t,\ P_2 = \frac{1}{2}(3t^2 - 1),\ P_4 = \frac{1}{2}(5t^3 - 3t).$$

1.1 Homework

1. Find the first few 6 terms in the series solution for

$$x'' + \cos(t)x = 0$$

2. Solve

$$x'' + tx' + x = 0$$

with $x(0) = 1$ and $x'(0) = 0$. What is the interval of convergence?

3. Use series to find the first 3 nonzero terms to the solution to

$$x' = 1 + tx^2 \quad x(0) = 0$$

Note that this is a nonlinear equation.

2 Singularities

Many interesting physical problems involve differential equations which do not have analytic coefficients as the cases above. The simplest such equation is the Cauchy-Euler equation:

$$t^2 x'' + tpx' + qx = 0.$$

If you try to solve this with series, it will not work as you can see that the coefficient involving x is actually q/t^2 which is singular at $t = 0$. There are several ways to solve this equation however. One is to change the independent variable letting $t = e^s$. (How did I know to do that? I knew since I knew what the solution would look like...) In this case (using the chain rule) we get

$$x_{ss} + (p - 1)x_s + qx = 0$$

and we can easily solve this. Let r_1, r_2 be the roots of the characteristic equation $r^2 + (p-1)r + q = 0$. If the roots are real and distinct, then $x(s) = e^{rs}$ or $x(t) = t^r$. If the roots are real and repeated, the other solution is se^{rs} or $x(t) = \ln(t)t^r$. Finally, for complex roots, $x(s) = e^{as}\cos bs$ and $x(s) = e^{as}\sin bs$ or in terms of the original variable, $x(t) = t^a \sin(b \ln t)$, $x(t) = t^a \cos(b \ln t)$. Note that all of these have singularities at the origin unless the roots are real and positive integers.

Lets turn to a general system

$$x'' + P(t)x' + Q(t)x = 0$$

We say that $t = 0$ is a regular singular point if $t^2Q(t)$ and $tP(t)$ are analytic at $t = 0$. If an equation has a regular singular point, we can write it in the form

$$t^2x'' + tp(t)x' + q(t) = 0 \tag{2}$$

where $t^2Q = q, t^2P = p$ and p, q are now analytic functions at $t = 0$. Once again, we write

$$p(t) = \sum_{k=0}^{\infty} p_k t^k \quad q(t) = \sum_{k=0}^{\infty} q_k t^k.$$

Suppose that the polynomial (called the *indicial* equation)

$$f(r) = r^2 + (p_0 - 1)r + q_0$$

has real roots, r_1, r_2 with $r_2 \leq r_1$. Then (2) has a solution of the form

$$x_1 = |t|^{r_1} \sum_{j=0}^{\infty} a_j(r_1)t^j, \quad t < 0 \quad \text{or} \quad t > 0$$

with $a_0 \neq 0$ and

$$f(r_1 + j)a_j(r_1) = -\sum_{k=0}^{j-1} [(k + r_1)p_{j-k} + q_{j-k}]a_k(r_1).$$

The series converges in the same interval as that of p, q and if $r_1 - r_2$ is not an integer, there is a second linearly independent solution

$$x_1 = |t|^{r_2} \sum_{j=0}^{\infty} a_j(r_2)t^j, \quad t < 0 \quad \text{or} \quad t > 0$$

where the coefficients are the same as above but r_2 replacing r_1.

This looks complicated, but if you proceed stepwise, it is not so bad.

2.1 Bessel's equation

Bessel functions are the main reason that I have introduced this. In many linear PDEs that involve cylindrical coordinates, Bessel's differential equation arises. This is

$$t^2x'' + tx' + (t^2 - p^2)x = 0.$$

Obviously, $t = 0$ is a regular singular point and the indicial equation is

$$r^2 - p^2 = 0.$$

The roots are $r = \pm p$. If p is an integer, then the procedure we have illustrated gives only one linearly independent solution. But lets just work with this solution for now. Substitute $x_1 = t^p \sum_0^\infty a_j t^j$ into this obtaining

$$t^2 \sum_{j=0}^{\infty} (j + p)(j + p - 1)a_j t^{j+p-2} \quad + \quad t\sum_{j=0}^{\infty} (j + p)a_j t^{j+p-1}$$
$$+ \quad t^2 \sum_{j=0}^{\infty} a_j t^j - p^2 \sum_{j=0}^{\infty} a_j t^j = 0$$

After some manipulation, this leads to

$$(1 + 2p)a_1 t + \sum_{j=2}^{\infty} [j(j + 2p)a_j + a_{j-2}]x^j = 0$$

giving us the desired recursion,

$$a_1 = 0 \quad a_j = -1\frac{a_{j-2}}{j(j+2p)}$$

which depends on the arbitrary constant a_0. If p is not an integer, then replacing p with $-p$ yields a second solution. If p is an integer, then clearly negative values of p will cause the recursion to become singular. Otherwise, the ratio test shows that this is analytic on the whole line. We cas write the series in a compact form by noting that the odd coefficients disappear ($a_1 = 0$) so that

$$x_1 = a_0 x^p \left\{ 1 + \sum_{j=1}^{\infty} \frac{(-1)^j t^{2j}}{2^{2j} j!(1+p)\cdots(j+p)} \right\}.$$

Suppose that $p = n$ is an integer and we choose $a_0 = 1/(2^n n!)$. Then we get the Bessel function of the first kind of integer order:

$$J_n(t) = \left(\frac{t}{2}\right)^n \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+n)!} \left(\frac{t}{2}\right)^(2j).$$

They looked like damped trig functions.

Bessel functions have many interesting properties.

If $r_1 = r_2$ in the indicial equation, then the second linearly independent solution is

$$x_2 = x_1 \ln|t| + |t|^{r_1} \sum_0^{\infty} c_j t^j$$

If $r_1 - r_2$ is a positive integer, then

$$x_2 = \alpha x_1 \ln|t| + |t|^{r_2}(1 + \sum_1^{\infty} d_j t^j)$$

where α is a possibly 0 constant.

As a last example, consider

$$t^2 x'' - tx = 0.$$

The indicial equation has roots 0 and 1. One solution is

$$x_1 = t \sum_{j=0}^{\infty} a_j t^j$$

and we find that

$$x_1 = \sum_{j=0}^{\infty} \frac{t^{j+1}}{(j+1)!j!}$$

Now we find the other solution. Set

$$x_2 = \alpha x_1 \ln|t| + \sum_{j=0}^{\infty} d_j t^j \quad d_0 = 1$$

We rewrite the equation in simpler form as $Lx = tx'' - x$. We use the fact that $Lx_1 = 0$ to get

$$0 = Lx_2 = \alpha x_1 L \ln|t| + L \sum_{j=0}^{\infty} d_j t^j$$

and applying the required differentiations:

$$0 = \alpha \sum_{j=0}^{\infty} \frac{2j+1}{(j+1)!j!} x^j + \sum_{j=0}^{\infty} [j(j+1)d_{j+1} - d_j] x^j.$$

This leads to

$$j(j+1)d_{j+1} - d_j = -\alpha \frac{2j+1}{(j+1)!j!}.$$

This gives e.g.

$$-d_0 = -\alpha \quad 2d_2 - d_1 = \frac{-3\alpha}{2} \quad 6d_3 - d_2 = \frac{-5\alpha}{12}$$

Since $d_0 = 1$ this determines α. The rest follow. The parameter d_1 is arbitrary so we set it to 0 for convenience.

2.2 Homework

Find series solutions to

1. $$t^2 x'' - t/2x' + tx = 0$$

2. Laguerre's equation:

   $$tx'' + (1-t)x' + px = 0$$

   Hint: The answer to this is in your book (problem 10, 11.4) Show that if p is a non-negative integer, then $x(t)$ is a polynomial. Assume that $x(0) = 1$ and find the first 5 such polynomials (that is $p = 0$, $p = 1$ up to $p = 5$.)

3. $$t^2 x'' - tx' + tx = 0$$

   (Note the indicial equation has roots that differ by an integer in this case.)