1 Regular series

We will study the solutions to certain second order linear differential equations. We will start with equations that are analytic at \( t = 0 \) and then turn to those that have singularities. Recall that a function is analytic at a point \( t = t_0 \) if it has a convergent Taylor series at that point. We are interested first in equations of the form:

\[
x'' + p(t)x' + q(t)x = 0
\]

where \( p(t), q(t) \) have Taylor series at \( t = 0 \). The basic idea is to assume that there is a solution of the form:

\[
x(t) = a_0 + a_1 t + a_2 t^2 + \ldots
\]

and then try to prove that the series converges. In general, we will get an iterative series of equations for \( a_j \) in terms of already known coefficients. The first two coefficients will be determined by the initial conditions:

\[
x(0) = a_0 \quad x'(0) = a_1.
\]

Let's do a simple example that we already can solve:

\[
x'' + x = 0
\]

Substituting the series into the derivative:

\[
x''(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + \ldots + j(j - 1)t^{j - 2}a_j + \ldots
\]

and equating the coefficients we get

\[
0 = a_0 + 2a_2 \\
0 = a_1 + 6a_3 \\
0 = a_2 + 12a_4 \\
0 = a_{2j - 1} + (2j + 1)(2j)a_{2j + 1} \\
0 = a_{2j} + (2j + 2)(2j + 1)a_{2j + 2}
\]

Thus we get a recursive relationship for the coefficients:

\[
a_{2j + 2} = - \frac{1}{(2j + 2)(2j + 1)} a_{2j} \\
a_{2j + 1} = - \frac{1}{(2j + 1)(2j)} a_{2j - 1}
\]

From this it is easy to see that

\[
a_{2j + 1} = (-1)^j \frac{a_1}{(2j + 1)!}
\]

and

\[
a_{2j} = (-1)^j \frac{a_0}{(2j)!}
\]

The series breaks into an even part that depends only on \( a_0 \) and an odd part that depends only on \( a_1 \). Writing the two series down, we immediately recognize them:

\[
x_{\text{even}}(t) = a_0 (1 - \frac{t^2}{2!} + \frac{t^4}{4!}) \ldots = a_0 \cos t
\]

and

\[
x_{\text{odd}}(t) = a_1 (t - \frac{t^3}{3!} + \frac{t^5}{5!}) \ldots = a_1 \sin t.
\]
Thus, we recover the already known solutions, \( \sin(t) \), \( \cos(t) \).

Let’s try an example that we don’t already know:

\[
x'' - tx = 0
\]

Once again we expand in a series and obtain

\[
\begin{align*}
2a_2 &= 0 \\
6a_3 - a_0 &= 0 \\
12a_4 - a_1 &= 0 \\
20a_5 - a_2 &= 0 \\
j(j - 1)a_j - a_{j-3} &= 0
\end{align*}
\]

This implies that

\[
a_j = \frac{1}{j(j - 1)} a_{j-3}.
\]

Note that \( a_2 = a_5 = a_8 \ldots = 0 \), \( a_3, a_6, \ldots, a_{3j} \) depend on \( a_0 \) and \( a_4, a_7, \ldots, a_{3j+1} \) depend on \( a_1 \). Note that the series converges for all \( t \) since

\[
\left| \frac{a_{j+3}t^{j+3}}{a_j t^j} \right| = |t|^3 \frac{1}{j(j - 1)}
\]

and this tends to zero as \( j \to \infty \) so by the ratio test the series converges for all \( t \). Some rearranging of terms yields

\[
\begin{align*}
a_{3j} &= \frac{1}{(3j)(3j - 1)} \frac{1}{(3j - 3)(3j - 4)} \ldots \frac{1}{3 \cdot 2} \cdot a_0 \\
a_{3j+1} &= \frac{1}{(3j + 1)(3j)} \frac{1}{(3j - 2)(3j - 3)} \ldots \frac{1}{4 \cdot 3} \cdot a_1.
\end{align*}
\]

This equation is called Airy’s equation and the two linearly independent solutions are called Airy functions.

In general we have the following result. Consider

\[
x'' + p(t)x' + q(t)x = 0 \quad x(0) = a_0 \quad x'(0) = a_1
\]

and suppose that

\[
p(t) = \sum_{k=0}^{\infty} p_k t^k, \quad q(t) = \sum_{k=0}^{\infty} q_k t^k
\]

converge in some interval \( J \) containing \( t = 0 \). Then

\[
x(t) = \sum_{j=0}^{\infty} a_j t^j
\]

where

\[
(j + 2)(j + 1)a_{j+2} + \sum_{k=0}^{j} p_{j-k} a_{k+1} (k + 1) + q_{j-k} a_k = 0
\]

and the series converges on the whole interval \( J \).

As a last application of regular series, we apply it to the Legendre’s equation which arises in the study of certain partial-differential equations defined in a spherical domain:

\[
(1 - t^2)x'' - 2tx' + p(p + 1)x = 0.
\]

We rewrite this as

\[
x'' = \frac{2t}{1 - t^2} y' + \frac{p(p + 1)}{1 - t^2}.
\]
We use the binomial expansion to expand the denominator

\[(1 - t^2)^{-1} = 1 + t^2 + t^4 + \ldots + t^{2j} + \ldots.\]

Note that this series converges only for \(|t| < 1\). Using the above recursion we get after some manipulation (I will leave it to you to verify this) that

\[a_{j+2} = -\frac{(p + j + 1)(p - j)}{(j + 2)(j + 1)}a_j.\]

This series converges from the ratio test as long as \(|t| < 1\). Furthermore note that the even terms depend only on \(a_0\) and the odd on \(a_1\). Note that if \(p\) is an integer, the series is finite and only goes out to \(p + 2\) terms. If \(p = n\) is an integer, we call the solutions the Legendre polynomials:

\[P_0 = 1, \ P_1 = t, \ P_2 = \frac{1}{2}(3t^2 - 1), \ P_4 = \frac{1}{2}(5t^4 - 3t).\]

1.1 Homework

1. Find the first few 6 terms in the series solution for

\[x'' + \cos(t)x = 0\]

2. Solve

\[x'' + tx' + x = 0\]

with \(x(0) = 1\) and \(x'(0) = 0\). What is the interval of convergence?

3. Use series to find the first 3 nonzero terms to the solution to

\[x' = 1 + tx^2, \ x(0) = 0\]

Note that this is a nonlinear equation.

2 Singularities

Many interesting physical problems involve differential equations which do not have analytic coefficients as the cases above. The simplest such equation is the Cauchy-Euler equation:

\[t^2x'' + tp'x' + qx = 0.\]

If you try to solve this with series, it will not work as you can see that the coefficient involving \(x\) is actually \(q/t^2\) which is singular at \(t = 0\). There are several ways to solve this equation however. One is to change the independent variable letting \(t = e^s\). (How did I know to do that? I knew since I knew what the solution would look like...) In this case (using the chain rule) we get

\[x_{ss} + (p - 1)x_s + qx = 0\]

and we can easily solve this. Let \(r_1, r_2\) be the roots of the characteristic equation \(r^2 + (p - 1)r + q = 0\). If the roots are real and distinct, then \(x(s) = e^{r_1s}\) or \(x(t) = t^{r_1}\). If the roots are real and repeated, the other solution is \(se^{r_1s}\) or \(x(t) = \ln(t)t^{r_1}\). Finally, for complex roots, \(x(s) = e^{r_1s}\cos bs\) and \(x(s) = e^{r_1s}\sin bs\) or in terms of the original variable, \(x(t) = t^a \sin(b \ln t), \ x(t) = t^a \cos(b \ln t)\). Note that all of these have singularities at the origin unless the roots are real and positive integers.

Let’s turn to a general system

\[x'' + P(t)x' + Q(t)x = 0\]
We say that \( t = 0 \) is a regular singular point if \( t^2Q(t) \) and \( tP(t) \) are analytic at \( t = 0 \). If an equation has a regular singular point, we can write it in the form

\[
t^2x'' + tp(t)x' + q(t) = 0
\]

where \( t^2Q = q, t^2P = p \) and \( p, q \) are now analytic functions at \( t = 0 \). Once again, we write

\[
p(t) = \sum_{k=0}^{\infty} p_k t^k \quad q(t) = \sum_{k=0}^{\infty} q_k t^k.
\]

Suppose that the polynomial (called the indicial equation)

\[
f(r) = r^2 + (p_0 - 1)r + q_0
\]

has real roots, \( r_1, r_2 \) with \( r_2 \leq r_1 \). Then (2) has a solution of the form

\[
x_1 = |t|^{r_1} \sum_{j=0}^{\infty} a_j(r_1)t^j, \quad t < 0 \text{ or } t > 0
\]

with \( a_0 \neq 0 \) and

\[
f(r_1 + j)a_j(r_1) = -\sum_{k=0}^{j-1} [(k + r_1)p_{j-k} + q_{j-k}]a_k(r_1).
\]

The series converges in the same interval as that of \( p, q \) and if \( r_1 - r_2 \) is not an integer, there is a second linearly independent solution

\[
x_1 = |t|^{r_2} \sum_{j=0}^{\infty} a_j(r_2)t^j, \quad t < 0 \text{ or } t > 0
\]

where the coefficients are the same as above but \( r_2 \) replacing \( r_1 \).

This looks complicated, but if you proceed stepwise, it is not so bad.

2.1 Bessel’s equation

Bessel functions are the main reason that I have introduced this. In many linear PDEs that involve cylindrical coordinates, Bessel’s differential equation arises. This is

\[
t^2x'' + tx' + (t^2 - p^2)x = 0.
\]

Obviously, \( t = 0 \) is a regular singular point and the indicial equation is

\[r^2 - p^2 = 0.\]

The roots are \( r = \pm p \). If \( p \) is an integer, then the procedure we have illustrated gives only one linearly independent solution. But lets just work with this solution for now. Substitute \( x_1 = t^p \sum_{j=0}^{\infty} a_j t^j \) into this obtaining

\[
t^2 \sum_{j=0}^{\infty} (j + p)(j + p - 1)a_j t^{j+p-2} + \sum_{j=0}^{\infty} (j + p)a_j t^{j+p-1} + t^2 \sum_{j=0}^{\infty} a_j t^j - p^2 \sum_{j=0}^{\infty} a_j t^j = 0
\]

After some manipulation, this leads to

\[
(1 + 2p)a_1 t + \sum_{j=2}^{\infty} [j(j + 2p)a_j + a_{j-2}]x^j = 0.
\]
giving us the desired recursion,
\[ a_1 = 0 \quad a_j = -1 \frac{a_{j-2}}{j(j + 2p)} \]

which depends on the arbitrary constant \( a_0 \). If \( p \) is not an integer, then replacing \( p \) with \(-p\) yields a second solution. If \( p \) is an integer, then clearly negative values of \( p \) will cause the recursion to become singular.

Otherwise, the ratio test shows that this is analytic on the whole line. We can write the series in a compact form by noting that the odd coefficients disappear \( a_1 = 0 \) so that

\[ x_1 = a_0 x^p \left\{ 1 + \sum_{j=1}^{\infty} \frac{(-1)^j t^{2j}}{2j j!(1 + p) \cdots (j + p)} \right\}. \]

Suppose that \( p = n \) is an integer and we choose \( a_0 = 1/(2^n n!) \). Then we get the Bessel function of the first kind of integer order:

\[ J_n(t) = \left( \frac{t}{2} \right)^n \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j + n)!} \left( \frac{1}{2} \right)^{2j}. \]

They looked like damped trig functions.

Bessel functions have many interesting properties.

If \( r_1 = r_2 \) in the indicial equation, then the second linearly independent solution is

\[ x_2 = x_1 \ln |t| + |t|^{r_1} \sum_{j=0}^{\infty} c_j t^j \]

If \( r_1 - r_2 \) is a positive integer, then

\[ x_2 = \alpha x_1 \ln |t| + |t|^{r_2} \left( 1 + \sum_{j=1}^{\infty} d_j t^j \right) \]

where \( \alpha \) is a possibly 0 constant.

As a last example, consider

\[ t^2 x'' - tx = 0. \]

The indicial equation has roots 0 and 1. One solution is

\[ x_1 = t \sum_{j=0}^{\infty} a_j t^j \]

and we find that

\[ x_1 = \sum_{j=0}^{\infty} \frac{t^{j+1}}{(j + 1)!j!} \]

Now we find the other solution. Set

\[ x_2 = \alpha x_1 \ln |t| + \sum_{j=0}^{\infty} d_j t^j \quad d_0 = 1 \]

We rewrite the equation in simpler form as \( Lx = tx'' - x \). We use the fact that \( Lx_1 = 0 \) to get

\[ 0 = Lx_2 = \alpha x_1 L \ln |t| + L \sum_{j=0}^{\infty} d_j t^j \]

and applying the required differentiations:

\[ 0 = \alpha \sum_{j=0}^{\infty} \frac{2j + 1}{(j + 1)!j!} x^j + \sum_{j=0}^{\infty} [j(j + 1)d_{j+1} - d_j] x^j. \]
This leads to
\[ j(j+1)d_{j+1} - d_j = -\alpha \frac{2j+1}{(j+1)!j!}. \]
This gives e.g.
\[ -d_0 = -\alpha, \quad 2d_1 - d_2 = -\frac{3\alpha}{2}, \quad 6d_3 - d_2 = -\frac{5\alpha}{12}. \]
Since \( d_0 = 1 \) this determines \( \alpha \). The rest follow. The parameter \( d_1 \) is arbitrary so we set it to 0 for convenience.

### 2.2 Homework

Find series solutions to

1. \[ t^2x'' - t/2x' + tx = 0 \]

2. Laguerre’s equation:
\[ tx'' + (1-t)x' + px = 0 \]

Hint: The answer to this is in your book (problem 10, 11.4) Show that if \( p \) is a non-negative integer, then \( x(t) \) is a polynomial. Assume that \( x(0) = 1 \) and find the first 5 such polynomials ( that is \( p = 0, p = 1 \) up to \( p = 5 \).

3. \[ t^2x'' - tx' + tx = 0 \]

(Note the indicial equation has roots that differ by an integer in this case.)