1. Consider the planar ODE:

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

such that $f_x + g_y = 0$ for all x, y in the plane. Prove that there is a function $H(x, y)$ such that $f(x, y) = H_y$ and $g(x, y) = -H_x$. Prove that the orbital derivative of $H(x, y)$ is zero. Prove that if (x_0, y_0) is a local minimum for $H(x, y)$, then it is a stable equilibrium point to the system of ODEs. Prove that the linearization always leads to pure imaginary eigenvalues in this case. Consider the equation $x' = y$, $y' = -f(x)$. Show $H(x, y) = y^2/2 + F(x)$ where $F(x)$ is the antiderivative of $f(x)$. By plotting $F(x)$, show that it is possible to sketch the phase portrait for the planar system by looking at minima and maxima of $F(x)$. In particular, show that the equilibria are saddle points for maxima of F and centers for minima of $F(x)$. Find $H(x, y)$ for the following systems

   (a) $x' = x^2 y + y^2$, $y' = -xy^2 + x^3$
   (b) $x' = q(y)$, $y' = -r(x)$
   (c) Sketch the phase-space picture for $x' = y$, $y' = x(1 - x^2)$.

2. Use $V = x^2 + y^2$ to analyze the following planar ODEs. What stability conclusions can be drawn.

   (a) $x' = -x^3 + 2y^3$, $y' = -2xy^2$
   (b) $x' = -x^3 + 2xy^2$,$y' = -2x^2 y - y^3$
   (c) $x' = y - x^3$, $y' = -x$
   (d) $x' = -y$,$y' = x + y^5 - 2y$

3. Analyze $y'' + f(y)y' + h(y)$ where $f(y) > 0$ and $yh(y) > 0$ for $y \neq 0$ and such that f, g are continuous. (Hint: convert this to a system of 2 odes and find a suitable Lyapunobv function.) Additionally, show that if

$$\lim_{|y| \to \infty} \int_0^y h(s)\, ds = +\infty$$

then all solutions to this ODE are bounded.

4. The Lorenz equation has the form:

$$\begin{aligned} x' &= \sigma(y - x) \\ y' &= rx - y - xz \\ z' &= xy - bz \end{aligned}$$

where σ, r, b are positive parameters. Show that $V = x^2 + \sigma y^2 + \sigma z^2$ is a strict Liapunov function for $r < 1$ and thus the origin is globally aymptotically stable.