HOMEWORK 1 - Math 2920 - ODE I ERMENTROUT and RUBIN

This homework is due at the start of class on Thursday, September 20th, 2012.

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1. Prove that
$$\phi(t, x) = \frac{1}{\frac{1}{x} - t}$$
is a dynamical system.

2. Problem 1.10, pg. 11, TESCHL. HINT: Assume that there are two distinct solutions and consider their difference.

3. Find all continuous solutions of the initial value problem
$$x' = x^{1/3}, \; x(0) = 0.$$
Be sure to provide an argument that your list is complete and to consider both $t < 0$ and $t > 0$.

4. Prove that if $x(t)$ solves the differential equation $\dot{x} = x^2 - t^2$ with $x(t_0) = t_0$, then there exists a time $t \geq t_0$ such that $x(t) < 0$.

5. Problem 1.30, pg. 28, TESCHL.

6. Consider the initial value problem
$$x' = x^2, \;\; x(0) = x_0 > 0. \tag{1}$$
The existence theorem guarantees that for $x' = f(x), x(0) = x_0$, a solution exists up to time $t = M/\beta$, where f is bounded by M for $|x - x_0| < \beta$.

   For a given x_0, find a choice of β to maximize the interval of existence you get from the theorem and compare it to the actual interval of existence that you get from the solution of equation (**??**).

7. Let $f(x) = Ax$, where A is a constant matrix. Show that each component of the n^{th} Picard iteration to any solution of $x' = f(x)$ is a polynomial of degree at most n.

8. Certain Southeast Asian fireflies flash at a natural frequency close to once per second. These insects tend to synchronize their flashes with each other, as well as with external light sources presented in experiments. If an external source is presented that flashes at a frequency near the natural frequency, the fireflies will adjust their flashing to this new frequency. However, if a source is presented that flashes too fast, then the phase of the firefly flashing drifts relative to this source.

   We can model the flashing of a firefly by using an angular variable $\theta \in [0, 2\pi]$, such that a flash occurs when $\theta = 0$, then θ increases until it reaches 2π, at which point θ is reset to 0 and another flash occurs. The corresponding ordinary differential equation (ODE) takes the form $d\theta/dt = \omega$, where ω is the frequency of the flashing (in units of 1/seconds) and we identify each even multiple of π with 0. Notice that this ODE has solution $\theta(t) = \omega t$.

When an external source that flashes at frequency ω_f is present, the relevant equation becomes

$$\frac{d\theta}{dt} = \omega + a\sin(\omega_f t - \theta) \tag{2}$$

for some positive constant a. By substituting $\theta = \omega t$ into the right hand side of (**??**), we see that if the external source flashes faster than the fireflies alone, then the sine term speeds up the fireflies, while if the source is slower, then it slows down the fireflies, as observed experimentally.

Now, define the phase difference between the fireflies and the source as $\phi(t) = \theta - \omega_f t$. We say that *entrainment* occurs when $\phi(t)$ tends to a constant value, independent of t, such that the fireflies always fire at a fixed phase relative to the source.

(a) Derive an ODE for $\phi(t)$. (The equation should include the first derivative of ϕ, a function of ϕ, and constant terms, but not θ.)

(b) Draw the phase line pictures, with stability of any fixed points indicated, for the two cases $0 < \omega - \omega_f < a$ and $\omega - \omega_f > a$.

(c) Explain why entrainment occurs for $0 < \omega - \omega_f < a$ and phase drift occurs for $\omega - \omega_f > a$.

(d) In the case that $\omega - \omega_f > a$, one can compute the time it takes for ϕ to increase from 0 to 2π, corresponding to a drift of firefly firings through all possible phase relations to the source. Compute this time, called the *beat period*, from the differential equation for ϕ.