**HOMEWORK 2 - Math 2920 - ODE I** ERMENTROUT & RUBIN - Fall 2012

This assignment is due in class on Tuesday, October 9, 2012.

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Let $f$ be a function from $U$, an open subset of $\mathbb{R}^n$, to $\mathbb{R}^n$, throughout this assignment.

1. Problem 2.6 (Teschl), pg. 39, parts (i) and (iii).

   In the next three problems, we consider three different cases where it can be shown that finite time blow-up does not occur:

2. Suppose that $f$ is Lipschitz on $U = \mathbf{R}^n$. Show that every solution to $x' = f(x)$ exists on $(-\infty, \infty)$; that is, no solutions blow up in finite time. (Hint: It may be useful to use the integral formulation of the equation and argue by contradiction.)

3. Suppose that $x(t) > 0$ and that

$$\frac{dx}{dt} \leq x^2$$
$$\int_0^\infty x(s)\, ds < \infty$$

   Prove that $x(t) < \infty$ for all $t$. (Note: this is not obvious since the integral constraint does not eliminate singularities like $1/\sqrt{t}$; here is a hint: write $x^2$ as $x(t)x(t)$ and think about how to solve $x' = a(t)x(t)$.)

4. Consider $x' = f(x)$ where $f : U \to \mathbb{R}^n$ is locally Lipschitz. Let us show that we can find a locally Lipschitz function $\alpha(x) : U \to \mathbb{R}$ such that the solutions to $x' = \alpha(x)f(x)$ exist on $(-\infty, \infty)$.

   a) Show that if $\alpha(x) : U \to \mathbb{R}$ and $f(x) : U \to \mathbb{R}^n$ are locally Lipschitz, then so is $\alpha(x)f(x)$.

   b) Show that if $f(x) : U \to \mathbb{R}^n$ is locally Lipschitz, then so is the function $\alpha(x) = \dfrac{1}{1 + |f(x)|}$.

   c) Show that, for $g(x) : U \to \mathbb{R}^n$ locally Lipschitz, if there exists $M > 0$ such that $|g(x)| \leq M$ for all $x \in U$, then the solutions to $x' = g(x)$ exist on $(-\infty, \infty)$. (Hint: The integral formulation may again be useful.)

   If we put together a) and b), then we see that $g(x) = \alpha(x)f(x) = \dfrac{f(x)}{1 + |f(x)|}$ is locally Lipschitz on $U$. Clearly it is bounded on $U$. Thus, by c), the solutions to $x' = \alpha(x)f(x)$ exist on $(-\infty, \infty)$, as claimed.

5. Exercise 4 (Grant), page 20.

6. Solve the integral equation:

$$x(t) = 1 + \int_0^t sx(s)\ ds.$$

Compute the first few terms for the Picard iteration (starting from x(0)=1) and then use this to write the general $n^{th}$ iterate. Show that the terms are same as the Taylor series expansion for your solution to the integral equation.

7. It is possible to prove existence directly using nothing more than calculus. Suppose that $f(t, x)$ is defined in $[t_0 - \alpha, t_0 + \alpha] \times B(\bar{a}, \beta)$ and bounded with bound, $M$. Let $b = \min(\alpha, \beta/M)$. $f$ has a Lipschitz constant $L$. Let

$$y_{n+1} = a + \int_{t_0}^t f(s, y_n(s))\ ds$$

with $y_0 = a$. Prove by induction that

$$|y_{n+1}(t) - y_n(t)| \leq \frac{ML^n(t - t_0)^{n+1}}{(n+1)!}$$

Conclude that

$$a + \sum_{n=0}^{\infty} [y_{n+1}(t) - y_n(t)] = y(t)$$

is uniformly convergent on $[t_0 - b, t_0 + b]$, that is

$$y(t) = \lim_{n \to \infty} y_n(t)$$

exists uniformly. Show that $y(t)$ is a solution to

$$y(t) = a + \int_{t_0}^t f(s, y(s))\ ds.$$

8. Show that the system

$$\frac{dy}{dx} = \cos x + xy^{1/3},\ y(0) = 0 \tag{1}$$

does not satisfy the conditions for the standard uniqueness theorem.

Prove that there exists a unique solution to equation (1) as follows.

(a) Assume that $f(x_0, y_0) \neq 0$. Show that the function $Y : I \subset \mathbb{R} \to \mathbb{R}$ is a solution of

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \tag{2}$$

on a neighborhood $N$ of $(x_0, y_0)$ on which $f(x, y) \neq 0$ if and only if $Y^{-1}$ is a solution of

$$\frac{dx}{dy} = \frac{1}{f(x, y)},\ x(y_0) = x_0$$

on $N$.

(b) Show that if $f(x, y)$ is Lipschitz in $x$ in $N$ as above, then equation (2) has a unique solution on some $N' \subset N$.

(c) Explain how part (b) yields the desired uniqueness of solutions to (1).

9. Problem 2.16 (Teschl), pg. 50.