HOMEWORK 3 - Math 2920 - ODE I RUBIN & ERMENTROUT - Fall 2012

This assignment is due in class on Thursday, October 25th, 2012.

---

1. Express e^{tA} as a polynomial in A for

$$A = \begin{bmatrix} -1 & -1 & -4 \\ -1 & -1 & -4 \\ -4 & -4 & 2 \end{bmatrix}$$

2. Sketch all the possible phase portaits of the system

$$x' = ax - 2y, \quad y' = 3x - y$$

as the parameter varies from $-\infty < a < \infty$.

3. Problem 3.26, pg. 85.

4. Problem 3.28, pg. 85.

5. Suppose that $A(t+T) = A(t)$ and that $A(-t) = -A(t)$. Let $\Phi(t)$ be the principal matrix for $x' = A(t)x$ with $t_0 = 0$ Show that $\Phi(-t) = \Phi(t)$. Use this result to show that all solutions to $x' = A(t)x$ are periodic.

6. Let $A(t) = A(t+T)$ be a 2×2 periodic matrix. The characteristic equation for the monodromy matrix can be written as

$$r^2 - ar + b.$$

Let T define the triangle in the (a, b) plane with vertices, $(-2, 1), (2, 1), (0, -1)$. Show that (a) all Floquet multipliers lie inside the unit circle if and only if (a, b) are inside the triangle; (b) there exist $T-$periodic solutions along the left edge; and (c) there exist $2T-$periodic solutions along the right edge. Also, determine (d) what kinds of solutions exist along the top edge?

7. In this problem and the next, we will study $x' = A(t)x$. As noted in class, it is possible that for the solution $x(t)$,

$$||x(t)|| \to \infty \text{ as } t \to \infty, \tag{1}$$

even though $A(t)$ has negative eigenvalues for each fixed t. Assume that $A(t)$ is a 2×2 matrix with real, distinct eigenvalues, $\lambda_1(t) < \lambda_2(t) < 0$ for each t.

(a) Let $\mathcal{B}$ denote the set of 2×2 matrices with eigenvalues of negative real part such that $x \cdot Bx > 0$ for some $x \in \mathbb{R}^2$. Prove that if (1) occurs, then $A(t) \in \mathcal{B}$ for some $t > 0$.

(b) Let $A = A(t_0)$ for some fixed t_0. Let $\delta \in (0, \pi)$ denote the angle between the eigenvector corresponding to λ_1 and that corresponding to λ_2. Show that there exists an invertible matrix Q such that

$$QAQ^{-1} = B := \begin{bmatrix} \lambda_1 & (\lambda_2 - \lambda_1)\cot(\delta) \\ 0 & \lambda_2 \end{bmatrix}.$$

(c) Let $r(x) = x \cdot Bx$ for $x \in \mathbb{R}^2$. Find the maxima of $r(x)$ on the unit circle. Prove that $r(x) > 0$ for some $x \in \mathbb{R}^2$ if and only if $\sin(\delta) < (1-p)/(1+p)$ for $p = \lambda_2/\lambda_1$; note that your proof should include an explanation of why it suffices to consider $r(x)$ on the unit circle.

(d) Plot $r(x)$ over the unit circle for the matrix $A(t)$ given by

$$A(t) = \begin{bmatrix} -1 - 9\cos^2(6t) + 12\sin(6t)\cos(6t) & 12\cos^2(6t) + 9\sin(6t)\cos(6t) \\ -12\sin^2(6t) + 9\sin(6t)\cos(6t) & -1 - 9\sin^2(6t) - 12\sin(6t)\cos(6t) \end{bmatrix}, \tag{2}$$

for a fixed t. (That is, for the t that you choose, let x vary around the unit circle and plot r as a function of the angle θ corresponding to each x.)

8. Here we build off of the previous exercise to see how to generate examples of 2×2 matrices $A(t)$ such that (1) occurs.

(a) Let $R(t, \omega)$ denote the rotation matrix

$$R(t, \omega) = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}.$$

Find a 2×2 matrix $G(\omega)$ such that $R(t, \omega) = e^{tG(\omega)}$.

(b) Set $A(t) = R(t, \omega)B[R(t, \omega)]^{-1}$ for $B \in \mathcal{B}$. Let $x' = A(t)x$ and define $y(t) = [R(t, \omega)]^{-1}x(t)$. Show that

$$y' = [B - G(\omega)]y$$

and find the corresponding expressions for $y(t), x(t)$, for any given $x(0)$.

(c) For the example (2), find B, ω, and the eigenvalues of $B - G(\omega)$.

(d) Based on the solution obtained in (b), to generate a 2×2 matrix $A(t)$ with negative real eigenvalues such that (1) holds, it suffices to find 2×2 matrices B and $G(\omega)$ such that there exists a nonzero vector v and a $\lambda > 0$ for which $[B - G(\omega)]v = \lambda v$. Prove that this requirement can be met if and only if $B \in \mathcal{B}$. Hint: As noted earlier, it suffices to consider v to be a unit vector. Think geometrically: What does it mean for $[B - G(\omega)]v = \lambda v$ to hold for $\lambda > 0$? What does it mean to have $B \in \mathcal{B}$? (For further details and related issues, see Josić and Rosenbaum, *SIAM Review*, 2008.)