MATH 2371: Homework # 9

1. Consider the $n \times n$ matrix, A defined as follows. For $2 < i < n-2$, $a_{ii} = -4$, $a_{i,i-1} = a_{i,i-2} = a_{i,i+1} = a_{i,i+2} = 1$. Additionally, $a_{11} = -2, a_{12} = 1, a_{13} = 1, a_{22} = -3, a_{23} = a_{24} = a_{21} = 1, a_{nn} = -2, a_{n,n-1} = a_{n,n-2} = 1$, and $a_{n-1,n-1} = -3, a_{n-1,n} = a_{n-1,n-2} = a_{n-1,n-3} = 1$.

   - Prove that A is noninvertible.
   - Bound the spectral radius and show that all the eigenvalues have non-positive real parts.
   - Suppose that $a_{11} = -3$. Prove that the matrix is invertible and that all the eigenvalues have strictly negative real parts.

2. Consider the polynomial:

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

   Show that the roots of $p(z)$ satisfy:

$$|z| \leq \max\{|a_0|, |a_1| + 1, \ldots, |a_{n-1}| + 1\}$$

   with strict inequality if all of these terms are not equal. Hint: Consider the companion matrix for $p(z)$ and use Corollary 6.28.

3. In corollary 6.2.8, show that irreducibility is necessary by providing an example in which there is at least one value of i so that $R_i < |||A|||_\infty$ but the spectral radius is not strictly less that $|||A|||_\infty$.

4. Show that an irreducible matrix cannot have a zero row or column.

5. Suppose that A is a positive matrix (that is, all entries are positive) and suppose that h is the normalized ($h_1 + \ldots h_n = 1$) eigenvector corresponding to the dominant eigenvalue. Prove that

$$\rho(A) = \sum_{i,j=1}^{n} a_{ij} h_j$$

6. If a positive matrix is nonsingular, show that the inverse cannot be nonnegative. If a nonnegative matrix, A (entries greater than or equal to 0) is nonsingular, prove that the inverse matrix can be nonnegative only if A has exactly one nonzero entry in each column. How is this related to a permutation matrix?