Assignment 6

1. Problems 2, 3, 5, 6 in Chapter 6

2. Suppose that $AB = BA$ where $A, B$ are square $n \times n$ matrices. Suppose that $A$ has distinct eigenvalues.

   (a) Show that if $h_j$ is an eigenvector for $A$ with eigenvalue, $a_j$, then the same $h_j$ is an eigenvector of $B$ with eigenvalue $b_j$. Thus, show that the matrix $H = [h_1, h_2, \ldots, h_n]$ satisfies:

   \[ AH = H \text{diag}(a_1, \ldots, a_n) \]

   and

   \[ BH = H \text{diag}(b_1, \ldots, b_n) \]

   (b) Now show that there is a polynomial, $q(x)$ of degree at most $n - 1$ such that $B = q(A)$. Here are some hints: Use the fact that

   \[ Pq(A)P^{-1} = q(PAP^{-1}) \]

   and use the proof of Theorem 6 on page 12!

3. For the following matrices, find an eigenbasis (generalized and genuine) and use this to either diagonalize them or put them in upper triangular form. Finally find the minimum polynomial for each.

   \[ A_1 = \begin{pmatrix} -3 & 2 & 2 \\ -1 & 1 & 1 \\ -3 & 2 & 2 \end{pmatrix} \]

   and

   \[ A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix} \]

4. Suppose that $A^2 = A$. Show that $A$ can be diagonalized. (Hint: what is the minimum polynomial for $A$?)

5. True or False? If a triangular matrix, $T$ is similar to a diagonal matrix, then $T$ must in fact be diagonal.

6. Find a $3 \times 3$ matrix whose minimum polynomial is $x^2$.

7. Let

   \[ A = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix} \]

   Prove that the characteristic polynomial for $A$ is

   \[ x^3 - ax^2 - bx - c \]

   and that this is also the minimum polynomial.
8. Let $A, B$ be defined as below. Show that they commute and thus find a matrix, $P$ such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix}.$$