1. A chemostat is a device that continuously pumps nutrient into a tank where bacteria use it to grow takes place and then they are removed for use. The equations for this are:

\[
S' = F(S_0 - S) - K_1 NS/(K_2 + S)
\]

\[
N' = -FN + \gamma K_1 NS/(K_2 + S)
\]

\(S\) has units of moles/liter, \(N\) has number of bacteria per liter. Time is in hours. Given this, (a) What are the dimensions of the parameters, \(F, S_0, K_1, K_2, \gamma\)? (b) By rescaling \(N, C, t\) you should be able to reduce this to only two dimensionless parameters. Start with letting \(N = An, S = Ns, t = CT\). There are at least two choices to make here, so I will force your hand. Let \(A = K_2\) and proceed from there and having made the correct scaling you should have:

\[
s' = (a - s) - ns/(1 + s)
\]

\[
n' = -n + bns/(1 + s)
\]

where, you should tell me what \(a, b\) are in terms of the original parameters. (c) Find the equilibria and determine their stability. (d) Assume that \(a > 1\) and \(b > 1/(a - 1)\) and sketch the nullclines in the \(s, n\) plane along with some representative trajectories. Use \(a = 2, b = 2\) for simplicity. (e) Now, for this part, you have an equilibrium, \(n\) in terms of the parameters, \(a, b\). Get \(N\) in terms of the original parameters. Now, the rate of bacteria production is \(FN\) (since that is what is collected) What value of \(F\) maximizes this rate?

2. (a) Consider the simple epidemiological model that we discussed in class:

\[
S' = -\beta SI + \delta (N - S - I)
\]

\[
I' = I(\beta S - \gamma)
\]

Find all the equilibria and stability of this. Sketch the phase plane and the nullclines for \(\beta N/\gamma > 1\). Use the phase-plane to sketch the solution to the differential equation given \(S(0) = N\) and \(I(0)\) is a small positive number (b) Compute the sensitivity of nonzero equilibrium for thr infecteds, \(I\) with respect to the contact rate, \(\beta\).

3. Recall from homework #1 that you fit some data to the logistic function by replotting numerical derivatives and using least squares on a linear set of equations. Here we you should do it using the way that we discussed in class. First, recall:

\[
N(t, a, K, N_0) = \frac{KN_0}{{\exp(-at)}[K - N_0] + N_0}
\]

(a) Compute the sensitivity of \(N(t)\) with respect to \(a\) and with respect to \(K\) the carrying capacity (b) Suppose that we know that \(K = 5\) and
$x_0 = 1$. We given $(t_j, N_j)$ data as $(t_1, N_1) = (2, 1.9)$, $(t_2, N_2) = (4, 3.5)$, $(t_3, N_3) = (6, 4)$ and $(t_4, N_4) = (8, 4.5)$. Compute the best fit for the parameter $a$ by minimizing:

$$\sum_{j=1}^{4} (N(t_j, a, 5, 1) - N_j)^2.$$  

4. Consider the following chemical reaction system:

$$\begin{align*}
2X &\rightarrow 3X \\
3X &\rightarrow 2X \\
X + Y &\rightarrow 2Y \\
Y &\rightarrow \star
\end{align*}$$

with the first three rate constants equal to 1 and the last, $d$. (a) Using the Laws of Mass Action write the differential equations for $x, y$. (b) Assume, $d$ and find the equilibria and their stability. You can ignore the $(x, y) = (0, 0)$. (c) Set $d = 0.75$ and draw the phase-plane with nullclines in the region $-0.05 < x < 1.1$ and $-0.05 < y < 1.1$. Sketch a few trajectories either using the computer (XPP code can be found on the website) or by hand. (d) Let $d = 0.55$, describe what happens for $(x(0), y(0)) = (1.0, 0.25)$ (e) do the same thing for $d = 0.4$. 

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