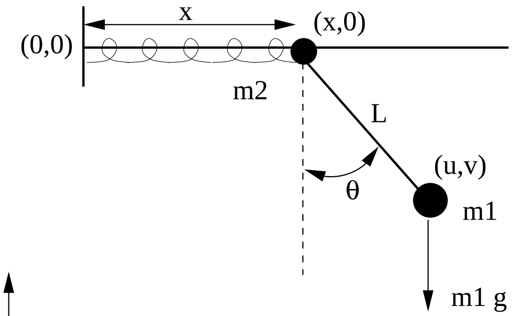

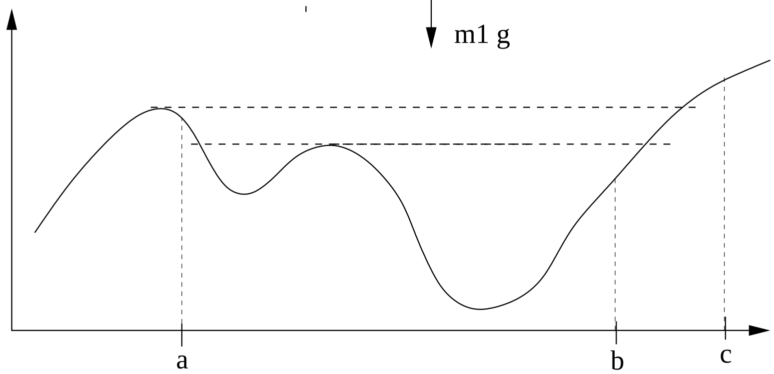

1. (20 pts) In the figure above, I depict a spring that is threaded through a rod with the left side attached to the wall. The right end of the spring is attached to a ball of mass m_2 that is free to move along the rod. Also attached to this ball is pendulum that can rotate freely in the plane. The mass of the bob of the pendulum is m_1. The spring has rest length a, and linear spring constant, k. There are two degrees of freedom: x, the distance of the spring from the wall, and θ, the angle of the pendulum with respect to the vertical axis.

   - (5 pts). Using the coordinate system given in the figure, show that the potential energy,

     $$P = (k/2)(x - a)^2 - m_1 g L \cos\theta$$

     and that the kinetic energy is

     $$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1/2)L^2\dot{\theta}^2 + m_1 L \dot{x}\dot{\theta}\cos\theta$$

     Write down the lagrangian, $B = T - P$.
   - (10 pts). Use the Euler-Lagrange equations to write down the differential equations for θ, x. They will be intermixed, that is there will be terms that mix $\ddot{\theta}, \ddot{x}$. You don't have to separate them out.
   - (5 pts).Find the equilibria for this model.
   - (Bonus - 5 pts) Simulate the model using the following parameters: $m_1 = m_2 = 1, k = 1, g = 1, a = 1, L = 2$. Start with $\theta = 0$ and $x = 2$. You will have to solve for $\ddot{x}, \ddot{\theta}$ for this.

2. (10 pts) At the bottom of the above figure, I have drawn a potential energy function, $F(x)$ for a one-degree of freedom system. The equations are

$$m\ddot{x} = -F'(x)$$

- Set $m = 1$ and sketch the phase plane in the $(x, \dot{x})$ phase plane and identify all saddle points, centers. Sketch relevant trajectories, including all the separatrices for the saddle points. Add arrows to indicate direction as well.
- Sketch the three trajectories such that $x(0) = x_0$ and $\dot{x}(0) = 0$ where x_0 corresponds to the three points, a,b,c, shown in your phase-plane. Also sketch $x(t)$ for enough time so I can see what is going on. Finally, identify which of the trajectories is periodic.

3. (25 pts) Consider the following chemical reaction

$$\begin{aligned} B + U &\longrightarrow V \\ U &\longrightarrow C \\ A &\longrightarrow U \\ 2U + V &\longrightarrow 3U \end{aligned}$$

where the four reactions have rates $k_{1,2,3,4}$ respectively.

- (5 pts) Assume that A, B are positive constants and write down the differential equations for U, V.
- (9 pts) Set $k_1 = k_2 = k_3 = k_4 = 1$. Find the equilibria and their stability as a function of A, B. What kind of behavior can you expect to happen when the equilibrium is unstable?
- (3 pts) When is the system a positive-feedback system; that is, the linearized matrix has the form

$$M = \begin{bmatrix} + & + \\ - & - \end{bmatrix}$$

- (8 pts) Set $A = 1, B = 1$ and draw the phase plane and the nullclines. Set $A = 1, B = 0.5$ and draw phaseplane and nullclines. Set $B = 3$ and simulate the system (with initial conditions, $U = 1, V = 0$) to help you draw the phaseplane. Which of these three cases is a positive feedback system?
- (Bonus 5 pts) Pick $A = 1, B = 1.5$ and consider the reaction-diffusion problem:

$$\begin{aligned} U_t &= F(U, V) + D_u U_{xx} \\ V_t &= G(U, V) + D_v V_{xx} \end{aligned}$$

where $0 < x < \pi$ and $U_x(0, t) = V_x(0, t) = U_x(\pi, t) = V_x(\pi, t) = 0$. Here F, G are the equations you derived above. What is the minimum ratio of D_u/D_v for which you could expect pattern formation?

4. (18 pts) On the small island of Tristan da Cunha (the most remote inhabited island in the world), where the population is about 300, the following data was found during a small outbreak of the common cold (brought to the island by people bringing supplies). The number of infecteds, $I(t)$ for days $t = 0 \ldots 10$ is $(1, 1, 3, 7, 6, 10, 13, 13, 14, 14, 17)$ and days $t = 11 \ldots 20$ is $(10, 6, 6, 4, 3, 1, 1, 1, 1, 0)$. We will model this as an SIR model

$$\begin{aligned} S' &= -\beta S I \\ I' &= \beta S I - \gamma I \\ R' &= \gamma I. \end{aligned}$$

Given $S(0) = 300, I(0) = 1, R(0) = 0$, one might try to pick β, γ so that a plot of $I(t)$ matches the data. In particular, the time of the peak and the value of the peak would be good to match. To do this requires more stuff that I did not cover in class, so I will make it easier. I used a fitting program and found $\gamma = 1.2$. You should simulate this equation and pick β to best fit the data. Given your value of β, plot $I(t)$ on the same plot as the data. Answer the following questions: (a) How many susceptibles are left once the cold has passed through? (b) Given your choice of β, what is the minimum population in order for this particular cold epidemic to take off? (c) When a real epidemic passes through, people tend to be more cautious. How would change, say, β to reflect the decreased contact rate as the epidemic progresses? (d) (Bonus for math junkies 5 pts). Prove that for any SIR model, once an epidemic passes through, the remaining susceptible population can never be re-infected by the same exact disease.

5. (25 pts) In this problem, you will explore several variants of the Hawk-Dove game. I introduce several new strategies. The first is the bully who threatens a dove but runs from a hawk. Here is his table with a dove and a hawk:

	B	H
B	(V-D)/2	0
H	V	(V-C)/2

	B	D
B	(V-D)/2	V
D	0	V/2

Here V is the payoff in territory and C is the cost for fighting like a hawk and D is the damage to the two bullies incur for fighting dirty. We assume V, D, C are nonnegative.

- (10 pts) Let x be the population of bullies and $1 - x$ be the dove or hawk. Write down the equations for the dynamics of the bully when paired with the hawk and with the dove. Determine which states are stable (pure bully,hawk,dove, or mixed) as a function of (V, C, D).
- (15 pts) Consider a three strategy game with hawks,doves, and bullies. Fill in the table, given that V is the gain from territory and C is the cost for fighting honorably and D for fighting dirty:

	B	H	D
B			
H			
D			

Let x be the number of bullies and y be the number of hawks with $1 - x - y$ the number of doves. Write down the equations for the dynamics of this system. Fix $V = 1$ and consider the following three examples ($C = .5, D = .75$), $C = 1.5, D = .75$, $C = 1.5, D = 2$. Draw the x, y phaseplane and nullclines in both cases. (I would use the computer for this). For each of the three cases, what are the fractions of Bullies, Hawks, and Doves?

6. (12 pts) (A) Develop a model for the following frightening scenario and write down the equations describing what each parameter is. There are three groups: humans (H), juvenile aliens (J), and adult aliens (A). All three groups have a natural death rate. There is a constant influx of new humans and juvenile aliens. Juvenile aliens grow up to be adult aliens. Humans kill juvenile aliens when they encounter them. Adult aliens kill humans when they encounter them. (B) Having written your model, now determine if it is qualitatively stable. If not explain why and if so, prove it. (C) Bonus. (5 pts) Show that there is a single positive equilibrium to your model. (You can do this without using much algebra at all, just using the intermediate value theorem that you learned in Calc 1.)

7. (25 pts) Here is a modification of the standard predator prey model:

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x) - \frac{xy}{x+a} \\
\frac{dy}{dt} &= y(-\mu + x)
\end{aligned}$$

Parameters μ, a are positive.

- (2 pts) Give an interpretation of the predation term $-xy/(a+x)$
- (8 pts) Find all the equilibrium points
- (8 pts) Determine their stability. In particular, show that if $\mu_1 < \mu < \mu_2$ then the predator y is nonzero and stable. You should find the two values $\mu_{1,2}$. Are there any Hopf bifurcations?
- (7 pts) Set $a = 1/4$. Sketch the phase plane for $\mu = 1/2$ and for $\mu = 1/4$. Include nullclines and if relevant any limit cycles. You do not have to use the computer for this but it may help.