LECT 2

Let's recall

$$X_{n+1} = A X_n$$

We want to know the fate of $X_n$. This is determined by the eigenvalues of $A$.

For $A$ 1x1, this is just $a$.

For $A$ 2x2,

$$\lambda^2 - \text{Tr}(A) \lambda + \det(A) = 0$$

$$\det A = 1$$

$$\text{Tr} A$$

$$(0, -1)$$

$$(-1, 1)$$

Complex

Real

Inside triangle $|\lambda| < 1$ outside triangle $|\lambda| > 1$

So we can always determine $\det A$.

Example 1: Romeo + Juliet
Let $R_n$ be Romeo's love or hate of Juliet on day $n$.

Let $J_n$ be Juliet's love or hate of Romeo on day $n$.

$R_n \Rightarrow$ Love $R_n < 0$ Hate $R_n = 0$ Neutral

$R_{n+1} = a_R R_n \quad J_{n+1} = a_J J_n$

Assume $a_R, a_J > 0$ (so don't have daily mood swings!)

$a > 1$ Then whatever it is it gets more passionate!

How do $R_n$ and $J_n$ interact?

Assume linear:

$R_{n+1} = a_R R_n + p_R J_n \quad R_0 = J_0 = 1$

$J_{n+1} = a_J J_n + p_J R_n$(both love each other)

4) Styles of romance

Easy to simulate

(i) $a_R = 0.5, \ a_J = 0.7 \quad p_R = 0.2 \quad p_J = 0.5$

E.g. $R_n, J_n$:

$R_n$ tends out!
\[ \text{Tr}(A) = a_n + a_j \quad \text{det}(A) = a_n a_j - p_n p_j \]

(i) \[ \text{Tr}(A) = 1.2 \quad \text{det}(A) = 1.35 - 1 = 0.35 \]

in the middle of stable, tripling so meters out

\[ |\lambda_j^n| \to 0 \]

(ii) \[ a_n = 1, \quad a_j = 1 \quad p_n = 0.2 \quad p_j = 0.2 \]

*Fickle* if Romeo hates Juliet then

*Juliet* increases love of Romeo + vice versa

\[ \text{Tr}(A) = 2 \quad \text{det}(A) = 1 + 0.4 = 1.4 \]

complex growing out of triangle.

(iii) \[ a_n = 0.5, \quad a_j = 0.8 \quad p_n = 0.2, \quad p_j = 0.5 \]

\[ \text{Tr} = 1.3 \quad \text{det} = 0.4 - 0.1 = 0.3 \]

\[ \text{det} = \text{Tr} - 1 \Rightarrow \lambda_1 = +1, \lambda_2 = -0.3 \]

\[ \lambda_1^n \to 1, \quad \lambda_2^n \to 0 \]
Will the plants survive?

\[ p_{n+1} = \alpha \sigma \gamma p_n + \beta \sigma (1-\alpha) s_n \]

\[ s_{n+1} = \sigma \gamma p_n \]

\[ A = \begin{bmatrix} \alpha \sigma \gamma & \beta \sigma (1-\alpha) \\ \sigma \gamma & 0 \end{bmatrix} \]

WANT TO BE OUT OF TRIANGLE

\[ \text{Tr} = \alpha \sigma \gamma \]

\[ \text{Det} = -\beta \sigma^2 \gamma (1-\alpha) \]

For example if \( \text{Det} < -1 \) then always out

\[ \Rightarrow \beta \sigma^2 \gamma (1-\alpha) > 1 \]

\[ \text{det } A < -\text{Tr}(A) - 1 \]

\[ < \text{Tr} A - 1 \]

Since \( \text{Tr}(A) < 0 \) (and \( \text{det} < -\text{Tr}(A) - 1 \))

\[ \Rightarrow \text{Det} < -\text{Tr}(A) - 1 \]

\[ \Rightarrow -\beta \sigma^2 \gamma (1-\alpha) < \alpha \sigma \gamma - 1 \]

Pretty easy

\[ \Rightarrow 1 < \gamma [\beta \sigma^2 (1-\alpha) + \sigma \Delta] \]

\[ \Rightarrow \gamma > \frac{1}{(\beta \sigma^2 (1-\alpha) + \sigma \Delta)} \frac{\exp h 3.5 \sigma}{10^3} \]
Nonlinear difference equation

We know that nothing can keep growing forever, so the equation for cells dividing, e.g.:

\[ M_{n+1} = a \cdot M_n \]

is probably only good for small if \( a > 1 \). Instead, there can be crowding or resource use to lead to:

\[ M_{n+1} = f(M_n) \cdot M_n \quad f(0) = 1 \]

\( f(M_n) \leq 1 \) is a function that accounts for slowing growth as \( M_n \) gets larger. The simplest \( f(M_n) = \alpha (1 - \frac{M_n}{K}) \)

\( K \) is called carrying capacity

\[ M_{n+1} = a \cdot (1 - \frac{M_n}{K}) \cdot M_n \]

Let \( X_n = \frac{M_n}{K} \) then \( \frac{M_{n+1}}{K} = a \cdot (1 - \frac{M_n}{K}) \cdot \frac{M_n}{K} \)

\[ X_{n+1} = a \cdot (1 - X_n) \cdot X_n \quad \text{WLOG}

"Logistic equation"

Ex 2 Fireflies, SE Asian Photuris Maura flashes rhythmically
Thousands of organisms synchronize!

If you flash a pulse of light \( T \) you will shift the time of the next flash.

\[
T' = T' (s)
\]

Define \( \Delta (s) = \frac{T - T'(s)}{T_0} \)

所谓相位重置曲线

For \( P, M \)

\[ \Delta (s) = -\alpha \sin \frac{2\pi s}{T_0} \]

Let flash light with period \( T \) + every what this does to the time of the flash.

Define phase, \( \theta_0 \) = time after last pulse called phase

\[ 0 \leq \theta < T \]. \text{ Let } \theta_n = \text{ phase at moment of light pulse}.

\[ \theta_{n+1} = \theta_n + 2 \mod T \]

With no synchrony P \& C. But:

\[
\theta_{n+1} = \theta_n + 1 - \alpha \sin \frac{2\pi \theta_n}{T}
\]

Nonlinear difference equation
Economic model with consumer sentiment

Let \( y(t) \) be income at time \( t \) (say month)

\[
D\ y(t-1) - y(t-2) \ni \text{Re income difference from previous period.}
\]

Let \( I(t) \) be income investment function

A consumption function

\[
C(y, D) = a + y\left(b + \frac{c}{1 + e^{-(y(t-2) - y(t-1))}}\right) - \frac{y(t-2) - y(t-1)}{1 + e^{-(y(t-2) - y(t-1))}} \rightarrow \infty \quad \text{sentiment} \rightarrow C
\]

\[
y(t-2) - y(t-1) \rightarrow \infty \quad \text{sentiment} \rightarrow C \Rightarrow \text{sent} \rightarrow 0
\]

Saving function

\[
d\ (y(t-2) - y(t-1)) + m \quad \text{Sentiment}
\]

\[
y(t) = a + d\ (y(t-2) - y(t-1)) + by(t-1) + I(y(t-1) - y(t-2)) + \frac{c\ y(t-1) + m}{1 + e^{-(y(t-2) - y(t-1))}}
\]

\( a = \text{autonomous expenditures like rent, food} \)

\( b = \text{trend toward consumption,\ } m = \text{trend toward saving} \)
The local pessimistic (less optimal spend) be heck!

\[ I(D) = \nu D - wD^3 \quad \text{e.g.} \]

Keynes model:

\[
y(t) = \rho (y(t-2) - y(t-1)) + a + b y(t-1)
\]

Hicks-Samuelson model:

\[
y(t) = (1 - \nu - s) y(t-1) - \nu y(t-2)
\]

Bolt linear.

General Nonlinear difference equation:

\[ x_{n+1} = F(x_n) \]

Equilibrium \( x_{n+1} = x_n \Rightarrow \)

\[ x = F(x) \]

Let \( x \) be equilibrium \( \bar{x} = F(x) \)

Want to know, e.g., is it stable.

\[ x_n = \bar{x} + y_n \quad x_{n+1} = \bar{x} + y_{n+1} = F(\bar{x} + y_n) \]
\[ F(X + Y) = F(X) + \bar{A}Y + \ldots \]

where \( \bar{A} \) is matrix of partial derivatives of \( F \) w.r.t. \( X \).

\[ Y_{n+1} = \bar{A}Y_n \]

\[ F = \left[ \begin{array}{c} f_1(x_1, \ldots, x_m) \\ \vdots \\ f_m(x_1, \ldots, x_m) \end{array} \right] \]

\[ \bar{A} = \left[ \begin{array}{ccc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \ldots & \frac{\partial f_m}{\partial x_m} \end{array} \right] \]

\[ (x_1, \ldots, x_m) = \hat{X} \]

called "Linearization."

\( \hat{X} \) is stable if all eigenvalues of \( \bar{A} \)

are stable (\( |\lambda| < 1 \)). Just like linear

Duffing Eqn.
First order difference equation

\[ x_{n+1} = f(x_n) \]

\[ x^* = f(x^*) \equiv \text{equilibrium} \]

\[ f'(x^*) = 0 \quad \text{if} \quad -1 < \beta < 1 \quad x^* \text{ unstable} \]

That’s why if we start near \( x^* \) we stay there and actually move to \( x^* \)

Cobwebbing

- Stretch \( f(x) \) vs \( x \)
- Draw \( y = x \)

Where \( y = x = y^* = f(x^*) \)
we have \( x = f(x) \)
A fixed pt.

\( x^* \) is stable!

\( x^* \) is unstable!
Analyze the Logistic equation

\[ x_{n+1} = ax_n(1-x_n) \]
\[ x^* = a x^*(1-x^*) \quad x^* = 0, \quad x^* = 1 - \frac{1}{a} \]
\[ f(x) = ax(1-x) \quad f'(x) = a - 2ax \]
\[ f'(0) = a \quad f'(1 - \frac{1}{a}) = a - 2a \left(1 - \frac{1}{a}\right) = 2 - a \]

Stability

\[ 1 + 1 \cdot |f'(x^*)| < 1 \]

\[ 1 < a < \frac{1}{2} \quad 0 \text{ is stable} \]

\[ 1 < 2 - a < -1 \Rightarrow 1 - \frac{1}{a} < 3 \Rightarrow 1 - a < 3 \]

\[ 0 \text{ is stable} \]

What happens for \( a \geq 3 \)?

We will look at this shortly!
Linear Diff Eqn Example

$x_{n+1} = x_n + x_{n-1}$  \[ x_0 = 0, \ x_1 = 1 \]

Fibonacci.

A General Solution is \( x_n = C \cdot 2^n \)

\[ -n+1 = 2^n + 2^{n-1} \Rightarrow 2 \cdot 2^{-n} = 2 \cdot 2 + 1 \]

\[ \Rightarrow 2^{-n} = \frac{1}{2} \pm \frac{\sqrt{5}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2} \]

(GOLDEN MEAN)

\[ x_n = A \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right)^n + B \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right)^n \]

\[ x_0 = A + B = 0 \Rightarrow A + B = 0 \Rightarrow A = -B \]

\[ x_1 = A \left[ \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right) - \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right) \right] = 1 \]

\[ \Rightarrow A = \frac{1}{\sqrt{5}} \]

\[ x_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right)^n - \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right)^n \right] \]

For \( n \) large \( \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right)^n \rightarrow 0 \)

So \( x_n \sim \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n \)

\[ \frac{x_{n+1}}{x_n} \sim \left( \frac{1 + \sqrt{5}}{2} \right) = 1.6180 \]

\( 8/5 = 1.6, \ \frac{5.5}{3.5} = 1.6176, \ \frac{89}{55} = 1.6181 \)
Remarks on Golden mean:

"Continued Fraction"

\[
- \frac{1}{X} = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}} \Rightarrow X = \frac{1}{1 + X} \Rightarrow X^2 + X - 1 = 0
\]

\[
x = \frac{-1 \pm \sqrt{5}}{2} \text{ since } x > 0
\]

Take the root

\[
\frac{1}{1 - x} = \frac{1}{1 - x} \Rightarrow X^2 - X + 1 = 0 \quad X = \frac{1 + \sqrt{5}}{2}
\]

\[
\text{Architecture}
\]

\[\text{Solve}\]

(a) \[X_n - 5X_{n-1} + 6X_{n-2} = 0 \quad X_0 = 2 \quad X_1 = 5
\]

\[
\lambda ^2 - 5 \lambda + 6 = 0 \quad (\lambda - 2)(\lambda - 3)
\]

\[
X = C_1 2^n + C_2 3^n \quad C_1 + C_2 = 2 \quad 2C_1 + 3C_2 = 5
\]

\[
\Rightarrow C_1 = C_2 = 1
\]

\[
X_n = 2^n + 3^n
\]

(b) \[X_n + \frac{1}{2}X_{n-1} + X_{n-2} = 0
\]

\[
\lambda ^2 + \frac{1}{2} \lambda + 1 = 0 \quad \lambda = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4}}{2} = -\frac{1}{2} \pm \frac{1}{2} \sqrt{15}
\]

\[
\lambda = e^{\pm (1.8234n) \pm i 5 \sin(1.8234n)}
\]