Sample Exam 1 - Math 1270

1. Solve the following initial value problems

(a) \( y' - 3y/t = t^2 \) with \( y(1) = 2 \)
(b) \( t^2y'' - 4ty' + 6y = 0 \) with \( y(1) = 0 \) and \( y'(1) = 1 \) (An euler equation, p166)
(c) \( y' = 1 + y^2 \) with \( y(0) = \alpha \). For each \( \alpha \) compute the interval of existence \( T_1 < t < T_2 \) for the solution.
(d) \( y' + 2y/t = y^3 \) with \( y(1) = 1 \). What is the interval of existence for the solution to this problem? (Note this is a Bernoulli equation, p77)
(e) \( y'' + 6y' + 8y = te^{-t} \) with \( y(0) = 0 \), \( y'(0) = 1 \).
(f) \( dy/dx = (x^2 + 3y^2)/(2xy) \) (see page 49-50)

2. Match the direction fields shown in the figure with the correct differential equation, \( y' = f(t, y) \). Here are the choices for \( f(t, y) \): (i) \( t - 1 \), (ii) \( 1 - y^2 \), (iii) \( y^2 - t^2 \), (iv) \( 1 - t \), (v) \( 1 - y \), (vi) \( t^2 - y^2 \), (vii) \( 1 + y \), and (viii) \( y^2 - 1 \). Explain your reasoning. Also, sketch a representative trajectory forward and backward in time with initial condition \( y(0) = 3/2 \). You do not need a computer to solve this.

3. Let \( f(y) = (y - 2)^2(y - 3)(y - 5)(y - 6) \). Find all the equilibrium points to \( y' = f(y) \) and determine their stability. In the \( (t, y) \) plane, sketch solutions with initial conditions \( y(0) = 1, 9, 2, 1, 3, 5, 5.5 \).

4. Solve the differential equation:

\[
(xy^2 - y)dy + (xy^2 - 2x^3 + x)dx = 0.
\]

5. If you attempt to tap your fingers in an alternating rhythm, as the frequency at which you tap increases, there is a sudden switch from alternate to synchronous tapping. Kelso and Haken proposed the following model for the phase-difference, \( \phi \), between your two fingers:

\[
\frac{d\phi}{dt} = -\sin \phi - \lambda \sin 2\phi.
\]

They suggest that as the frequency goes up, the parameter \( \lambda \) decreases. Find all the equilibria. Sketch the phase line when \( \lambda = 0, 1/2, 1 \), and \( \lambda = 1 \) indicating all stable and unstable equilibria in the interval \([0, 2\pi]\).

6. Consider the differential equation

\[
y'' + p(t)y' + q(t)y
\]

with continuous coefficients on an interval \((a, b)\). Let \( y_1(t), y_2(t) \) be two nonzero solutions such that they both have a local maximum at \( t = t_0 \in (a, b) \). Show that they do not form a fundamental set of solutions and that one is linearly related to the other.