Sample Exam 1

1. Sketch the phase line for the system \( \dot{x} = x^3(1-x^3) \). Sketch trajectories in the \( x-t \) plane for solutions such that \( X(0) = \left\{ -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\} \). Label all critical points and their stability. Sketch \( -\infty < t < \infty \).

2. Solve the following differential equation:
   a. \( y' + 2ty = 3t e^{-t^2} \) \( y(0) = 1 \)
   b. \( \frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy} \)
   c. \( (2x + 2y') dx + (3y^2 + 4xy) dy = 0 \)
   d. \( y'' + 2y' - 3y = 0 \) \( y(0) = 1, y'(0) = 0 \)
   e. \( y'' - 2y' + 5y = 0 \)
   f. For Problem (d), suppose \( y(0) = 1 \). Choose \( y'(0) \) so that \( y(t) \to 0 \) as \( t \to \infty \). Then choose it so \( y(t) \to 0 \) as \( t \to -\infty \).

3. A home buyer can afford no more than 800 \$/month + pay 9.7% interest on a 20 year loan. Assuming continuous payment of interest, what is the maximum loan he can get?

4. Prob # 39, page 156. Also prove that \( z_1(t) = \mathbf{y}_1(t) \)

5. Let \( y' = t^3, y(1) = 1 \) Find maximum interval of existence.
Answers
\[ x = x^2 (1 - x^2) = x^2 (1 - x)(1 + x) \]

2(a) \[ y' + 2ty = 3te^{-t^2} \]
\[ \ln f + \text{act} = e^{\frac{t^2}{2}} \]
\[ (ye^{t^2})' = 3te^{-t^2} e^{t^2} \]
\[ ye^{t^2} = \frac{3t^2 + c}{2} \]
\[ y(0) = 1 \Rightarrow c = 1 \]
\[ y = e^{-t^2} + \frac{3}{2} t^2 \]

(b) \[ \frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy} \]
Homogeneous. Let \[ v = \frac{y}{x}, y = xv \]
\[ v + x \frac{dv}{dx} = \frac{1 + 3v^2}{2v} \Rightarrow x \frac{dv}{dx} = \frac{1 + 3v^2 - v^2}{2v} = \frac{1 + v^2}{2v} \Rightarrow \]
\[ dv \left( \frac{2v}{1 + v^2} \right) = \frac{dx}{x} \Rightarrow \ln (1 + v^2) = \ln (x) + c \]
\[ \Rightarrow (1 + v^2) = cx \Rightarrow x^2 + v^2 = cx^3 \]
\[ x^2 + y^2 = cx^3 \]

(c) \[ (2x + 2y^2)dx + (3y^2 + xy)dy = 0 \]
\[ \frac{2x}{2y} = \frac{dy}{dx} = y = \text{exact} \]
\[ \theta = x^2 + 2y^2 x + \Psi(y) \]
\[ \Phi_y = 4yx + \Psi'(y) = 4xy + 3y^2 \Rightarrow \Psi = y^3 \]
\[ x^2 + 2y^2 x + y^3 = C \]
\[ d \ y'' + 2y' - 3y = 0 \quad r^2 + 2r - 3 = 0 \]
\[ r = -3, \ 2 \Rightarrow y = C_1 e^t + C_2 e^{-3t} \]
\[ y(0) = C_1 + C_2 = 1 \Rightarrow C_2 = \frac{1}{4}, \ C_1 = \frac{3}{4} \]
\[ y'(0) = C_1 - 3C_2 = 0 \]
\[ y(t) = \frac{3}{4} e^t + \frac{1}{4} e^{-3t} \]

(c) \[ r^2 - 2r + 5 = 0 \Rightarrow r = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i \]
\[ y = C_1 e^t \cos 2t + C_2 e^t \sin 2t \]

(f) \[ y = C_1 e^t + C_2 e^{-3t} \]
\[ y(0) = 1, \ y'(0) = a, \ y(t) \to 0 \ as \ y(0) = 1 \Rightarrow y'(0) = -3 \]
\[ y(t) \to 0 \ as \ t \to +\infty \Rightarrow C_2 = 0 \]
\[ \Rightarrow y = e^t \Rightarrow y'(0) = 1 \]

3) Interest rate: \[ \frac{9\%}{12} = 0.0075/\text{Month} \]
\[ \frac{dx}{dt} = -0.0075x - 8.00 \quad x(24) = 0, \ x(0) = ? \]
\[ x(t) = 10.6666.66 + Ce^{-0.0075 t} \]
\[ x(24) = 10.6666.66 + Ce^{-0.0075 \times 240} \]
\[ \Rightarrow C = 17,631.88 \]
\[ \Rightarrow x(0) = 10.6666.66 - 17,631.88 = 89,844.78 \]
(4) Since $y_1'(t_0) = y_2'(t_0) = 0$ at $t_0 \Rightarrow W[y_1, y_2](t)$ is zero $\Rightarrow$ not a P.S.
Suppose $y_1(t_0) = \alpha, y_2(t_0) = \beta$
Let $k = \beta/\alpha$
y_2(t) = $\frac{\beta}{\alpha} y_1(t)$. Since $y_2(t_0) = \beta, y_2'(t_0) = 0$
uniqueness Thm $\Rightarrow y_2(t) = ky_1(t)$

(5) $y' = y^3 + y(1) = 1$
$\frac{dy}{y^3} = dt \Rightarrow -\frac{1}{2} y^{-2} = \frac{1}{2} + c$
$-\frac{1}{2} = \frac{1}{2} + c \Rightarrow c = -1$
$\Rightarrow -\frac{1}{2} y^{-2} = \frac{t^2}{2} - 1 \Rightarrow y^2 = 2 - t^2$
\[ -\sqrt{2} < t < \sqrt{2} \]