The Theory of Linear ODEs

Math 1270

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1 First order systems.

Consider the homogeneous linear system:

$$\dot{x} = a(t)x \qquad x(t_0) = x_0 \tag{1}$$

This is readily solved by separation of variables

$$\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} a(s)ds$$

yielding:

$$x(t) = x_0 \exp\left[\int_{t_0}^{t} a(s)ds\right]$$

For example, the solution to

$$\dot{x} = -\sin(t)x \qquad x(0) = 1$$

is just $x(t) = \exp(\cos(t) - 1)$.

Now let's suppose that we want to solve

$$\dot{x} = a(t)x + b(t) \qquad x(t_0) = x_0. \tag{2}$$

There are several strategies; I will give you the most general. Recall that the solution to the homogeneous system was $x(t) = x_0 \exp(A(t))$ where $A(t)$ is the integral of $a(t)$. Consider

$$y(t) = x(t)\exp(-A(t)) \tag{3}$$

If we differentiate this with respect to time we see that

$$\dot{y} = (\dot{x} - a(t)x)\exp(-A(t)).$$

Rearrange (2) as

$$\dot{x} - a(t)x = b(t),$$

and multiply both sides be $\exp(-A(t))$. This yields

$$\dot{y} = (\dot{x} - a(t)x)\exp(-A(t)) = \exp(-A(t))b(t).$$

Now it is easy to solve for $y(t)$ since we just have

$$\frac{dy}{dt} = \exp(-A(t))b(t)$$

which has a solution:

$$y(t) = C + \int^{t} \exp(-A(s))b(s)ds.$$

Finally we use the definition of y to solve for x:

$$x(t) = e^{A(t)} \left[C + \int^t e^{-A(s)} b(s) ds \right]. \tag{4}$$

We now can solve any first order linear equation. The function $e^{-A(t)}$ is called the **integrating factor** for the system.

Here is an example.

$$\dot{x} = -x + \sin(t) \qquad x(0) = 0.$$

Here $a(t) = -1$ so $A(t) = -t$ so that the integrating factor is e^t. Multiplying both sides by this we get

$$\dot{y} = e^t \sin(t)$$

so that

$$y(t) = C + e^t(\sin t - \cos t)/2.$$

Thus the general solution is $x(t) = e^{-t} y(t)$ or

$$x(t) = Ce^{-t} + (\sin t - \cos t)/2.$$

The initial conditions are $x(0) = 0$ so $C = 1/2$ and we get

$$x(t) = \frac{1}{2}(\sin t - \cos t + e^{-t}).$$

Here is another example.

$$\dot{x} = -\frac{x}{t} + t^2 \quad x(1) = 1.$$

Here $a(t) = -1/t$ so that $A(t) = -\ln t$. The integrating factor is $e^{-A(t)} = t$ so multiplying this by t yields

$$\dot{y} = t^3$$

so that $y(t) = C + t^4/4$ and

$$x(t) = e^{A(t)} y(t) = (C + t^4/4)/t.$$

The initial condition implies that $C = 3/4$ so that

$$x(t) = \frac{3}{4t} + \frac{t^3}{4}.$$

1.1 Homework.

Solve the following first order equations

1. $\dot{x} = 2x + e^{2t}$ with $x(0) = 0$.
2. $\dot{x} = -tx + t^3$ with $x(0) = 1$.
3. $\dot{x} = 2x/(1 + t) + 1$ with $x(0) = 0$.

2 n^{th} order systems.

Here we consider constant coefficient linear systems of the form:

$$x^{(n)} + a_{n-1}x^{(n-1)} + \dots a_2 x'' + a_1 x' + a_0 x = 0. \tag{5}$$

We will often write this as

$$P(D)x(t) = 0$$

where $D = \frac{d}{dt}$ is the differentiation operator and $P(u)$ is the polynomial with coefficients given in (5.) There are two important things to consider. First, from the qualitative point of view, we'd like to know if the solutions to this decay as $t \to \infty$ or what they do over the long haul. The second thing is more quantitative and we want to know what the general solution to this equation is. As with linear algebraic equations, there are n linearly independent solutions to (5). These are readily found if we know the roots of the polynomial $P(u)$ called the **characteristic** polynomial. There are 4 cases of interest. Let r be a root of $P(u) = 0$. Then

- r **is real and distinct.** Then $x(t) = e^{rt}$ is a solution.
- $r = \alpha + i\omega$ **and is distinct.** Then there are two linearly independent solutions:

  $$x_1(t) = e^{\alpha t} \cos \omega t, \quad x_1(t) = e^{\alpha t} \sin \omega t.$$

- r **is real and repeated** $k > 1$ **times.** Then

  $$x_1(t) = e^{rt}, x_2(t) = te^{rt}, \dots, x_k(t) = t^{k-1}e^{rt}$$

  are the k linearly independent solutions.
- $r = \alpha + i\omega$ **and repeated** k **times.** Then there are $2k$ linearly independent solutions

  $$t^j e^{\alpha t} \cos \omega t, \; t^j e^{\alpha t} \sin \omega t \quad j = 0, \dots k-1.$$

Note that complex roots appear in pairs so that you only need to consider the one with a positive imaginary part.

Thus, if we can factor the polynomial, then we can find all the linearly independent solutions to the corresponding differential equation.

Here is an example. Find the general solution to

$$x'' = -x$$

The polynomial is $P(u) = u^2 + 1$ so that $r = \pm i$ are the roots. Thus, the two linearly independent solutions are $\cos t$ and $\sin t$ and the general solution is $x(t) = A \cos t + B \sin t$.

Here is another example. Find the general solution to

$$(D+1)^2(D^2 + 4D + 5)x = 0$$

The polynomial has -1 as a double root and $-2 \pm i$ as a pair of complex roots. Thus the general solution is

$$x(t) = Ae^{-t} + Bte^{-t} + Ce^{-2t} \cos t + De^{-2t} \sin t.$$

2.1 Homework

Solve the following linear ODEs

1. $x''' - 9x' = 0$
2. $x'''' + 2x'' - x = 0$
3. $x'' - 4x' + 4x = 0$

2.2 Stability of n^{th} order ODEs.

In many cases you are interested, not in the actual solutions, but whether or not the zero solution is stable for a system such as (5). There are two types of stability for linear systems and since our goal is to use this information for nonlinear systems, the only kind of stability we are interested in is exponential or asymptotic stability. That is, we demand that all solutions decay exponentially fast to zero as t increases. It is clear then, from the discussion above, that we want the real parts of all of the roots of the polynomial $P(u)$ to be negative. Then each of the linearly independent solutions to (5) will decay exponentially fast. Furthermore, as we will see later in the course, bifurcations occur when systems switch from being stable to unstable, so the stability and not the actual solutions is sometimes more important.

Fortunately, there is a way to determine whether the sign of the real part of the roots of a polynomial is always negative. Before diving into the general case, lets first consider $n = 1$:

$$x' + a_0 x = 0.$$

In this case $P(u) = u + a_0$ and the root of this is $-a_0$. If $a_0 > 0$ then all solutions decay exponentially while if $a_0 < 0$ they grow. Thus the condition is simply that $a_0 > 0$ implies all solutions decay.

Now, how about the general second order case:

$$x'' + a_1 x' + a_0 x = 0.$$

This is more difficult. We can use the quadratic formula to find the roots:

$$r_1 = \frac{1}{2}(-a_1 + \sqrt{a_1^2 - 4a_0}), \; r_2 = \frac{1}{2}(-a_1 - \sqrt{a_1^2 - 4a_0}).$$

If $a_0 < 0$ then both roots are real and since the product of the roots is just a_0 that means that one will be positive and one will be negative. So a_0 had better not be negative. If $a_0 > 0$ then there are two possibilities, the roots are real or they are complex. If they are complex, then a_1 better be positive. If they are real, then since the product of the roots is positive, they both are the same sign. The sum of the roots is $-a_1$ so that again a_1 better be positive. Thus, the condition for stability is that both a_1 and a_0 are positive. This sure is a lot easier than finding the roots. You can tell at a glance!

Let's now consider the general monic polynomial:

$$P(u) = u^n + a_{n-1}u^{n-1} + \dots a_1 u + a_0.$$

Form a series of matrices:

$$\begin{aligned}
H_1 &= a_{n-1} \\
H_2 &= \begin{bmatrix} a_{n-1} & 1 \\ a_{n-3} & a_{n-2} \end{bmatrix} \\
H_3 &= \begin{bmatrix} a_{n-1} & 1 & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-5} & a_{n-4} & a_{n-3} \end{bmatrix}
\end{aligned}$$

up to H_n. Note that each one is square and the first column contains every other coefficient, $a_{n-1}, a_{n-3}, a_{n-5} \dots$. Any index that is negative is replaced by 0 and 1 is always the entry that follows a_{n-1} with the remainder of the row 0's. Then, the roots of $P(u) = 0$ have negative real parts if and only if

$$\det H_j > 0 \quad j = 1 \dots n.$$

Remarks.

- $\det H_n = a_0 \det H_{n-1}$ so that this means that $a_0 > 0$. If $a_0 = 0$ then there is a zero root.
- if $\det H_{n-1} = 0$ but all the other criteria hold, then there are imaginary eigenvalues.

As a simple example, take $n = 2$, that is $P(u) = u^2 + a_1 u + a_0$.

$$\begin{aligned} H_1 &= a_1 \\ H_2 &= \begin{bmatrix} a_1 & 1 \\ 0 & a_0 \end{bmatrix} \end{aligned}$$

We see

$$\det H_1 = a_1, \quad \det H_2 = a_0 a_1$$

so the condition for stability is $a_1 > 0$ and $a_0 > 0$. Note that the remarks above imply that there is a zero root if $a_0 = 0$ and there are imaginary roots if $a_1 = 0, a_0 > 0$.

Let's look at $n = 3$ which we could not easily do in the way that we did $n = 2$ by finding the actual roots. The three matrices are

$$\begin{aligned} H_1 &= a_2 \\ H_2 &= \begin{bmatrix} a_2 & 1 \\ a_0 & a_1 \end{bmatrix} \\ H_3 &= \begin{bmatrix} a_2 & 1 & 0 \\ a_0 & a_1 & a_2 \\ 0 & 0 & a_0 \end{bmatrix} \end{aligned}$$

These have determinants

$$\det H_1 = a_2, \quad \det H_2 = a_1 a_2 - a_0, \quad \det H_3 = a_0(a_1 a_2 - a_0).$$

Thus, we have the result that the differential equation:

$$x''' + a_2 x'' + a_1 x' + a_0 = 0$$

has only exponentially decaying solutions if and only if

$$a_2 > 0, \quad a_0 > 0, \quad a_1 a_2 - a_0 > 0.$$

Furthermore, if $a_1 a_2 - a_0 = 0$ and a_2, a_0 are positive then there are imaginary roots according to the above remarks. So, at a glance we can see that all solutions to

$$x''' + x'' + 2x' + x = 0$$

decay exponentially but that some solutions to

$$x''' + x'' + 2x' + 3x = 0$$

grow exponentially and that there are periodic solutions to

$$x''' + x'' + 2x' + 2x = 0.$$

2.2.1 Homework

1. For the third order polynomial:
$$u^3 + a_2 u^2 + a_1 u + a_2 a_1 = 0$$
where $a_1, a_2 > 0$ we know that there are imaginary roots, $\pm i\omega$. Find them in terms of a_1, a_2. (Hint: substitute $i\omega$ into the above and set the real and imaginary parts to zero.) Finally, find the periodic solutions to
$$x''' + 2x'' + 2x' + 4x = 0.$$

2. For what value of the parameter r do all solutions to

$$x''' + rx'' + rx' + rx = 0$$

decay exponentially?

3. Find a set of conditions for the roots of

$$u^4 + a_3u^3 + a_2u^2 + a_1u + a_0 = 0$$

to have negative real parts and use your result to determine whether or not all solutions to

$$x'''' + 3x''' + 2x'' + x' + 2x = 0$$

decay exponentially?

3 Linear systems and the matrix exponential.

We now turn our attention to the system of equations:

$$\frac{dX}{dt} = A(t)X(t) \tag{6}$$

where $A(t)$ is an $n \times n$ dimensional matrix of continuous functions of time. Consider the set of solutions to (6), $X_j(t)$ satisfying

$$X_1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad X_2(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad X_n(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Since the equation is linear, any solution to (6) can be written as a linear combination of these n linearly independent solutions. Each one is initially one of the columns of the identity matrix. Now define the following matrix function:

$$\Phi(t) = [X_1(t)\ X_2(t)\ \ldots\ X_n(t)].$$

Each column of this matrix satisfies the differential equation and also $\Phi(0) = I$ is the identity matrix. This is called the **fundamental matrix** of solutions. Suppose that we want to solve (6) with initial data $X(0) = X_0$. Then the solution is just $X(t) = \Phi(t)X_0$. Thus, the fundamental matrix makes it easy to solve the initial value problem. How about the inhomogeneous problem:

$$\frac{dX}{dt} = A(t)X(t) + B(t)?$$

Before doing this, we state an important result:

$$\det\Phi(t) = \exp\left(\int_0^t \mathrm{Tr}A(s)\ ds\right). \tag{7}$$

That is, the determinant of the fundamental matrix is just an exponential of the integral of the trace of $A(t)$. This means that the inverse of the fundamental matrix always exists since the determinant is never zero.

Let $X(t) = \Phi(t)Y(t)$. Then

$$\frac{dX}{dt} = \frac{d\Phi}{dt}Y + \Phi\frac{dY}{dt} = A(t)\Phi Y + B(t)$$

But, $\frac{d\Phi}{dt} = A(t)\Phi$ so that we get

$$\Phi(t)\frac{dY}{dt} = B(t).$$

Thus

$$\frac{dY}{dt} = \Phi^{-1}(t)B(t)$$

and we can solve for $Y(t)$

$$Y(t) = Y(0) + \int_0^t \Phi^{-1}(s)B(s)\ ds.$$

Finally, this yields

$$X(t) = \Phi(t)\left[X(0) + \int_0^t \Phi^{-1}(s)B(s)\ ds\right]$$

as the solution to the inhomogeneous problem!

Here is an example. Solve the following system of equations:

$$\frac{dX}{dt} = \begin{bmatrix} -2t & -2t \\ 2t & 2t \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The fundamental matrix for $A(t)$ is

$$\Phi(t) = \begin{bmatrix} 1-t^2 & -t^2 \\ t^2 & 1+t^2 \end{bmatrix}$$

and the inverse is

$$\Phi^{-1}(t) = \begin{bmatrix} 1+t^2 & t^2 \\ -t^2 & 1-t^2 \end{bmatrix}$$

From

$$\int_0^t \Phi^{-1}(s)B(s)\ ds = \int_0^t \begin{bmatrix} s^2 \\ 1-s^2 \end{bmatrix} = \begin{bmatrix} t^3/3 \\ t-t^3/3 \end{bmatrix}$$

So,

$$X(t) = \frac{1}{3}\begin{bmatrix} t^3-3t^2 \\ 3-t^3+3t^2 \end{bmatrix}$$

Note that in general, we cannot find the Fundamental matrix for time-dependent systems; this example was cooked up.

3.1 Computing Φ for constant coefficient matrices.

In practice, it is usually impossible to compute $\Phi(t)$ for time-dependent matrices, $A(t)$. However, in the case of constant coefficients, it is easy. Consider the matrix function

$$E(t) = I + tA + \frac{t^2}{2!}A^2 + \ldots \frac{t^j}{j!}A^j + \ldots.$$

It is easy to prove that this series converges. As you probably recognize, this is the Taylor expansion for the exponential function. For this reason, the matrix, $E(t)$ is called the exponential of A and we write

$$E(t) = e^{At}.$$

Notice that $E(0) = I$ and if we differentiate,

$$E'(t) = A + tA^2 + \frac{t^2}{2!}A^3 + \ldots = AE(t)$$

so that in fact, $E(t)$ *is the fundamental matrix* for the constant coefficient linear system.

Here is an example. Take

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Note that

$$A^2 = -I, \quad A^3 = -A, \quad A^4 = A$$

so that the powers of A repeat every 4 times. Thus,

$$E(t) = \begin{bmatrix} 1 - t^2/2! + t^4/4! \ldots & t - t^3/3! + t^5/5! \ldots \\ -t + t^3/3! - t^5/5! \ldots & 1 - t^2/2! + t^4/4! \ldots \end{bmatrix}$$

which you recognize as

$$E(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

There must be an easier way, right - you can't really be expected to actually sum these series. In fact there is a really slick way to find the matrix exponential and thus the fundamental solution. Here is the method; it uses stuff from the previous section. Let A be an $n \times n$ constant matrix. The idea is that if we differentieate $\exp(At)$ with respect to t k times and evaluate at $t = 0$ then we can get a series of linear equations whose solution yields the fundamental matrix.

Here is the algorithm called "Fulmer's" method:

1. Write down the characteristic polynomial of A,

$$P(\lambda) = \det\ (\lambda I - A)$$

2. Consider the linear differential equation:

$$P(D)x = 0$$

and write down the n linearly independent solutions, $x_1(t), x_2(t), \ldots, x_n(t)$.

3. Now, we assert that the fundamental matrix is some matrix linear combination of these n solutions:

$$e^{At} = x_1(t)E_1 + x_2(t)E_2 + \ldots + x_n(t)E_n$$

where E_j are unknown $n \times n$ matrices. Set $t = 0$ in the above equation, differentiate and set $t = 0$ again, and so on up to $n - 1$ derivatives. This leads to the following equations:

$$\begin{aligned} I &= x_1(0)E_1 + x_2(0)E_2 + \ldots + x_n(0)E_n \\ A &= x_1'(0)E_1 + x_2'(0)E_2 + \ldots + x_n'(0)E_n \\ A^2 &= x_1''(0)E_1 + x_2''(0)E_2 + \ldots + x_n''(0)E_n \\ \ldots &= \ldots \\ A^{n-1} &= x_1^{(n-1)}(0)E_1 + x_2^{(n-1)}(0)E_2 + \ldots + x_n^{(n-1)}(0)E_n \end{aligned}$$

Solve the set of equations for E_j in terms of powers of A and you are done.

Here is an example. Suppose that A is 3×3 and the characteristic polynomial is $\lambda^2(\lambda - 1)$ Then the three linearly independent solutions are $1, t, e^t$ so we get

$$e^{At} = E_1 + tE_2 + e^t E_3.$$

Thus,

$$\begin{aligned} I &= E_1 + 0E_2 + E_3 \\ A &= E_2 + E_3 \\ A^2 &= E_3 \end{aligned}$$

From this we see that

$$E_3 = A^2, \quad E_2 = A - A^2, \quad E_1 = I - A^2$$

and thus

$$e^{At} = (I - A^2) + t(A - A^2) + e^t A^2.$$

We can apply this to the example above.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The characteristic polynomial is $\lambda^2 + 1$ so the roots are $\pm i$ and thus the two linearly independent solutions are $\cos t, \sin t$. So

$$e^{At} = E_1 \cos t + E_2 \sin t$$

and we find that

$$I = E_1 \quad A = E_2$$

so that

$$e^{At} = \Phi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

No series to sum at all!

3.2 Homework

1. Prove (7) for the 2×2 case. Here is how to proceed. Let

$$\Phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t). \end{bmatrix}$$

   Let

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t). \end{bmatrix}$$

   Differentiate the determinant of $\Phi(t)$, $\phi_{11}\phi_{22} - \phi_{21}\phi_{12}$ and use the fact that $\Phi' = A\Phi$ to show that

$$\frac{d}{dt}\text{det }(\Phi(t)) = (a_{11}(t) + a_{22}(t))\text{det}\Phi(t)$$

   and thus conclude the result (recall the solution to a first order homogeneous linear equation.)

2. Prove

$$\frac{d\Phi^{-1}(t)}{dt} = -A(t)\Phi^{-1}(t)$$

   (Hint: Differentiate $\Phi^{-1}(t)\Phi(t) = I$ with respect to t and use the fact that $d/dt\Phi(t) = A(t)\Phi(t)$.

3. Sum the series to find the fundamental matrix for

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

4. Find the matrix exponential for the following systems

   -

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$$

- $$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

- $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -9 & -1 \end{bmatrix}$$

4 Classification of planar linear systems.

Consider a system:

$$x' = ax + by, \quad y' = cx + dy.$$

There is obviously a fixed point at $(0,0)$ and if the matrix of coefficients is nonsingular, then it is the only fixed point. I now want to summarize the behavior in the phase-plane (x,y) in keeping with our qualitative methods. This is covered in much greater detail in Chapter 3 of our book; I will merely summarize the results.

The characteristic polynomial for the 2×2 matrix, A is just $P(\lambda) = \lambda^2 + (a+d)\lambda + (ad - bc)$. The coefficients are just the negative trace, $-T$ and the determinant D of the matrix A. The previous results thus imply that the fixed point $(0,0)$ is asymptotically stable if and only if $T < 0, D > 0$. Recall that the eigenvalues are

$$\lambda = \frac{1}{2}(T \pm \sqrt{\Delta}), \quad \Delta = T^2 - 4D,$$

where the quantity, Δ is called the discriminant. We can now classify the equilibrium further.

1. D<0. The determinant is negative implies that there are two real eigenvalues, one is positive and one is negative. We call the fixed point $(0,0)$ a **saddle point** and it is unstable.
2. D>0. The determinant is positive. There are two cases
   (a) $\Delta < 0$ implies that the eigenvalues are complex conjugates.
      i. $T < 0$, the trace is negative and we call the fixed point a **stable vortex** or **stable spiral.**
      ii. $T > 0$, the trace is positive and we call the fixed point an **unstable vortex** or **unstable spiral.**
      iii. $T = 0$, the trace vanishes and we call the fixed point a **center.**
   (b) $\Delta > 0$ implise the eigenvalues are both real and have the same sign.
      i. $T < 0$, the trace is negative and we call the fixed point a **stable node.**
      ii. $T > 0$, the trace is positive and we call the fixed point an**unstable node.**

Stable nodes and vortices are also called **sinks** while unstable nodes and vortices are called **saddles.** If the determinant is zero, this is a degenerate case and when the discriminant vanishes, this means there is a double eigenvalue and the node is either stable or unstable according to the sign of the trace.

4.1 Homework

1. Classify the following systems where (a,b,c,d) are given: (i) (1,0,0,-1); (ii) (1,-2,2,1); (iii) (1,-2,2,-1); (iv) (1,0,1,2); (v)(-1,2,2,-1); (vi) (r,2,-2,-1) as a function of r.
2. Sketch the phase-plane of these systems.