Exam 1 - Solutions

1. An arrow is shot into the air at a velocity of 10 m/sec, at a height of 2 meters, and an angle of $30^\circ$. What is the maximum height that it will reach?

   **SOL.** $a_y = -9.8 m/s$, so $v_y = v_y(0) - 9.8t$. $v_y(0) = 10 \sin 30 = 5 m/s$, so $v_y = 10 - 9.8t$. Note the max height occurs when $v_y = 0$, or $t = 5/9.8 = .51 s$. $r_y = r_y(0) + 5t - 4.9t^2$, so max height is 3.27 m.

2. Let $a = < 1, 2, 0 >$, $b = <-1, -1, 3 >$, and $c = < 2, -4, 1 >$.

   (a) What is $a \cdot (b - 3c)$?

   **SOL.** $b - 3c = <-7, 11, 0 >$, so $a \cdot (b - 3c) = -7 + 22 + 0 = 15$.

   (b) What is the equation for the plane formed by the vectors $a$ and $b$?

   **SOL.** Cross product is orthogonal to the plane, so

   $$a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ -1 & -1 & 3 \end{vmatrix} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$$

   So, $6x - 3y + z = 0$ is eqn for the plane.

   (c) Find the scalar component of $a$ onto $c$, that is $\text{comp}_c a$.

   **SOL.**

   $$\text{comp}_c a = \frac{a \cdot c}{|c|} = \frac{-8 + 0}{\sqrt{4 + 16 + 1}} = -\frac{6}{\sqrt{21}}.$$  

3. Let $r(t) = < t^2, t, t^3 >$.

   (a) What is the unit tangent vector at $t = 1$?

   **SOL.** $r' = < 2t, 1, 3t^2 >$, $r'(1) = < 2, 1, 3 >$, $T = r'(1)/|r'(1)| = < 2, 1, 3 > / \sqrt{14}$.

   (b) What is the curvature at $t = 1$?

   **SOL.** $\kappa = |r' \times r''|/|r'|^3$. $r'' = < 2, 0, 6t >$. $r''(1) = < 2, 0, 6 >$

   $$r' \times r'' = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 2 & 0 & 6 \end{vmatrix} = \begin{pmatrix} 6 \\ -6 \\ -2 \end{pmatrix}.$$  

   $$\kappa = \sqrt{36 + 36 + 4}/\sqrt{14}^3 = \sqrt{76}/\sqrt{14}^3.$$  

   (c) Find an equation for the plane normal to the curve at $t = 1$.

   **SOL.** Normal plane is perpendicular to the tangent! At $t = 1$, $r = < 1, 1, 1 >$, so from part a: $2(x - 1) + 1(y - 1) + 3(z - 1) = 0$, or, $2x + y + 3z = 6$.

   (d) Express the arclength of the curve from the point $< 0, 0, 0 >$ to $< 4, 2, 8 >$ as an integral (you don’t have to evaluate it.)

   **SOL.** At $t = 0$, $r = < 0, 0, 0 >$ and at $t = 2$, $r = < 4, 2, 8 >$. $ds = |r'(t)|dt$ so

   $$s = \int_0^2 \sqrt{4t^2 + 1 + 9t^4} dt.$$  

4. A projectile with acceleration $a(t) = < \sin(t), \cos(t), -4 >$ starts at $r(0) = < 1, 1, 1 >$ with an initial velocity of $v(0) = < 0, 0, 3 >$. 

(a) Find its position as a function of time.

**SOL.**  
\[ v(t) = \int a(t) + c_1, \quad r(t) = \int v(t) + c_2. \]  
Thus \( v(t) = \langle -\cos t, \sin t, -4t \rangle + c_1 \). At \( t = 0, < 0, 0, 3 >= < -1, 0, 0 > + c_1 \), so \( c_1 = < 1, 0, 3 > \) and \( v(t) = \langle 1 - \cos t, \sin t, 3 - 4t \rangle \).  
\[ r(t) = \langle t - \sin t, -\cos t, 3t - 2t^2 \rangle > + c_2. \]  
At \( t = 0, < 1, 1, 1 > = < 0, -1, 0 > + c_2 \), so, \( c_2 = < 1, 2, 1 > > \) and  
\[ r(t) = \langle 1 + t - \sin t, 2 - \cos t, 1 + 3t - 2t^2 \rangle > . \]

(b) At what point in time does the projectile reverse its direction in the \( z \)-direction? How high up is it at this point?

**SOL.** Reverses \( z \)-direction when \( z \)-velocity is zero, so that \( 3 - 4t = 0 \) or \( t = 3/4 \). The height is \( z_{\text{max}} = 1 + 3(3/4) - 2(3/4)^2 = 17/8 \).

5. Find the limits or prove that they do not exist.

(a)  
\[ \lim_{(x,y) \to (1,1)} \frac{x^2 - 3y^2}{x^3 + y^3} \]

**SOL.** Rational functions are continuous and the denominator is not zero when at \( (1,1) \), so limit is found by putting in the numbers, \( L = (1 - 3)/(1 + 1) = -1. \)

(b)  
\[ \lim_{(x,y) \to (0,0)} \frac{x^3 - y^3}{x^2 + y^2} \]

**SOL.** This is \( 0/0 \) so we have to be careful. Let \( x = r \cos \theta, y = r \sin \theta \) so that this becomes  
\[ \lim_{r \to 0} \frac{r^3 [\cos^3 \theta - \sin^3 \theta]}{r^2} = \lim_{r \to 0} r [\cos^3 \theta - \sin^3 \theta] \to 0 \]

(c)  
\[ \lim_{(x,y) \to (0,0)} \frac{x^3 y - xy^3}{(x^2 + y^2)^2} \]

**SOL.** Does not exist. Set \( x = ky \) and plug in to get  
\[ \frac{k^3 - k}{(k^2 + 1)^2} \]

which has many values.

6. Find a parametric equation for the surface of the lower half of the ellipsoid, \( 2x^2 + 4y^2 + z^2 = 1 \) (that is, the part with \( z \leq 0 \)).

**SOL.** Simple solution, \( x = x, y = y, z = -\sqrt{1 - 2x^2 - 4y^2} \) along with \( 1 - 2x^2 - 4y^2 \geq 0 \). A better way, \( x = (1/\sqrt{2}) \sin \theta \cos \phi, y = (1/2) \cos \theta \cos \phi \) and \( z = \sin \phi \) with \( 0 \leq \theta \leq 2\pi \) and \( \pi \leq \phi \leq 2\pi \).

7. Find the magnitude of the torque around the point \( P \) given the diagram below.

![Diagram of the torque problem](attachment:image.png)
**SOL.** Angle between $r$ and $F$ is 45, so $|\tau| = |F||r|\sin \theta = (50)(0.2)(\sin 45) = 5\sqrt{2}$ Nm.

8. Let $f(x, y) = x^3 + x^2y - y^2 - x - y^3$.
   (a) Find $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$.
   **SOL.** $f_x = 3x^2 + 2xy - y^2$, $f_y = x^2 - 2gx - 3y^3$, $f_{xx} = 6x + 2y$, $f_{yy} = -2x - 6y$, $f_{xy} = 2x - 2y$.
   (b) Find the linear approximation to $f(x, y)$ at the point $(1, 1)$ and use this to estimate the value of $f(x, y)$ at $(.9, .9)$.
   **SOL.** $z \approx L(x, y) := f(1, 1) + f_x(1, 1)(x-1) + f_y(1, 1)(y-1) = 0 + 4(x-1) - 4(y-1) = 4(x-y)$.
   $L(.9, .9) = 0$ so $f(.9, .9) \approx 0$.

9. A boat is pulled by two ropes as shown in the picture. Given a force of 255 N is needed to move the boat, find the magnitude of the force on each rope.

<table>
<thead>
<tr>
<th>20°</th>
<th>30°</th>
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<tbody>
<tr>
<td>255 N</td>
<td></td>
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**SOL:** $|T_1| \sin 20 = |T_2| \sin 30$. $|T_1| \cos 20 + |T_2| \cos 30 = 255$. Thus, $|T_2| = |T_1| \sin 20/\sin 30$. Thus,

$$|T_1| \left( \cos 20 + \frac{\cos 30 \sin 20}{\sin 30} \right) = 255$$

so $|T_1| = 166.439$ N and $|T_2| = 113.85$ N.

10. Match the contours with the surfaces.
   **SOL:** A1, B6, C2, D5, E4, F3