Tests for Convergence of Series

1. Geometric Series Test - Series Converges if $|r| < 1$.

   (a) $\sum_{n=2}^{\infty} 5 \cdot 3^{n+1} \cdot 2^{1-2n} = \sum_{n=2}^{\infty} 5 \cdot 3 \cdot 3^n \cdot 2 \cdot 2^{-2n} = \sum_{n=2}^{\infty} \frac{30 \cdot 3^n}{2^{2n}} = \sum_{n=2}^{\infty} \frac{30 \cdot 3^n}{4^n} = \sum_{n=2}^{\infty} 30 \left(\frac{3}{4}\right)^n$
   Since $|r| = 3/4 < 1$, this series will converge and the limit is

   $$\frac{1^{\text{st}}\text{term}}{1 - \text{ratio}} = \frac{135/2}{1 - 3/4} = 170$$

   (b) $\sum_{n=2}^{\infty} 5 \cdot 3^{n+1} \cdot 2^{-n} = \sum_{n=2}^{\infty} 5 \cdot 3 \cdot 3^n \cdot 2^{-n} = \sum_{n=2}^{\infty} 15 \left(\frac{3}{2}\right)^n$
   Since $|r| = 3/2 > 1$, this series will diverge.

2. Integral Test - Series Converges if Integral Converges

   (a) $\sum_{n=1}^{\infty} \frac{4}{1+n^2} \overset{\text{behaves-like}}{\approx} \int_1^{\infty} \frac{4}{1+x^2}\, dx = \lim_{t \to \infty} \int_1^t \frac{4}{1+x^2}\, dx = 4 \arctan x \Big|_1^{t \to \infty} = \pi$
   Therefore, since the integral converges, the series converges.

   (b) $\sum_{n=1}^{\infty} \frac{4}{1+n} \overset{\text{behaves-like}}{\approx} \int_1^{\infty} \frac{4}{1+x}\, dx = \lim_{t \to \infty} \int_1^t \frac{4}{1+x}\, dx = 4 \ln(1+x) \Big|_1^{t \to \infty} = \infty$
   Therefore, since the integral diverges, the series diverges.

3. Alternating Series Test - Series Converges if alternating and $b_n \to 0$.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ The series is alternating and $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0$.
   Therefore, the series converges.

   (b) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4n+1}$ The series is alternating but $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{n}{4n+1} = \frac{1}{4}$.
   Therefore, the series diverges.

4. Ratio Test - Series Converges if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$

(a) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ This would be too much work to use the integral test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3} < 1.$$

Since the absolute value of the ratio of consecutive terms in the series is always less than a value less than 1, like the geometric series, this series will converge.

(b) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ The factorial makes it impossible to use any other test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$$

Since the absolute value of the ratio of consecutive terms in the series is greater than a value of 1, this series will diverge.

(c) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ The factorial makes it impossible to use any other test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(10)^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \lim_{n\to\infty} \frac{10}{n+1} = 0$$

Since the absolute value of the ratio of consecutive terms in the series is less than 1, like the geometric series, this series will converge.

Definition: A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

(a) $\sum_{n=1}^{\infty} \frac{(-3)^n}{4^n}$ This series is absolutely convergent since $\sum_{n=1}^{\infty} \frac{3^n}{4^n}$ converges.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ This series is convergent but not absolutely convergent since the series of absolute values $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.