1. Determine whether the sum converges or diverges. Then state and show which convergence/divergence test you used.

   (a) $\sum_{n=1}^{\infty} \frac{4n}{5n+1}$
   Answer: Diverges since the limit of the terms is not zero.

   (b) $\sum_{n=0}^{\infty} \frac{4n}{n^2+n+1}$

   Answer: Diverges by Comparison test to $\sum_{n=0}^{\infty} \frac{4n}{n^2+1}$ which by the Integral test diverges because $\lim_{n\to\infty} 2\ln(n^2+1) = \infty$.

   (c) $\sum_{n=0}^{\infty} \frac{4}{n^2+3n+2}$
   Answer: Converges. Use partial fractions and take limit of partial sums.

   (d) $\sum_{n=0}^{\infty} \frac{2\cdot 7^n}{5\cdot 3^{2n}}$

   Answer: Converges. Ration is $\frac{7}{9} < 1$.

   (e) $\sum_{n=0}^{\infty} \frac{4}{n\cdot 2^n}$

   Answer: Converges. Use Ratio test: $\lim_{k\to\infty} \left| \frac{4}{(k+1)(2^{k+1})} \cdot \frac{k\cdot 2^k}{4} \right| = \frac{1}{2}$.

   (f) $\sum_{n=0}^{\infty} \frac{4}{n}$
   Answer: Diverges. This is the Harmonic Series. Also can use the Integral test.

   (g) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

   Answer: Converges. Use integral test: $-e^{1/x}\Big|_{1}^{x=t\to\infty} = -1+e$.

   (h) $\sum_{n=0}^{\infty} \frac{4^n}{3n!}$

   Answer: Converges. Use Ratio test: $\lim_{k\to\infty} \left| \frac{4^{k+1}}{3(k+1)!} \cdot \frac{3k!}{4^k} \right| = 0 < 1$.

   (i) $\sum_{n=0}^{\infty} \frac{3n!}{4^n}$

Answer: Diverges. Use Ratio test: $\lim_{k\to\infty}\left|\frac{3(k+1)!}{4^{k+1}}\cdot\frac{4^k}{3k!}\right| = \infty.$

(j) $\sum_{n=1}^{\infty}\frac{(-1)^n}{5n}$

Answer: Converges. Alternating Harmonic Series whose limit of terms goes to zero.

(k) $\sum_{n=1}^{\infty}\frac{(-1)^n 3n}{5n}$

Answer: Diverges. Alternating but the limit of the terms is not zero; it is $\frac{3}{5}$.

2. Evaluate the follow sum.

(a) $\sum_{n=0}^{\infty}\frac{2}{5}\left(\frac{-4}{7}\right)^n$

$S = \frac{2/5}{1-(-4/7)} = \frac{14}{55}.$

(b) $\sum_{n=0}^{\infty}\frac{2\cdot 7^n}{5\cdot 3^{2n}}$

$S = \frac{2/5}{1-(7/9)} = \frac{9}{5}.$

(c) $\sum_{n=0}^{\infty}\frac{6\cdot 3^n}{n!}$

$S = 6\sum_{n=0}^{\infty}\frac{3^n}{n!} = 6e^3.$

(d) $\sum_{n=0}^{\infty}\frac{4}{n^2+5n+4}$

$S_k = \frac{4}{3}\sum_{n=0}^{n=k}\left(\frac{1}{n+1}-\frac{1}{n+4}\right)$ and $S = \lim_{k\to\infty} S_k = \frac{22}{9}$ since:

$S_k = \frac{4}{3}\left(\left(1-\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{3}-\frac{1}{6}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\ldots+\left(\frac{1}{k+1}-\frac{1}{k+4}\right)\right).$

(e) $\sum_{n=0}^{\infty}\frac{(-1)^n}{5n!}$

$S = \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!} = \frac{1}{5e}.$

(f) $\displaystyle\sum_{n=2}^{\infty} 6\left(\frac{7}{8}\right)^n$

$$S = \frac{6(49/64)}{1-7/8} = \frac{147}{4}.$$

(g) $\displaystyle\sum_{n=1}^{\infty} 6\left(\frac{9}{10}\right)^n$

$$S = \frac{6(9/10)}{1-9/10} = 54.$$

3. Determine the radius and interval of convergence of the series given.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n}$

$$\lim_{n\to\infty}\left|\frac{(2x-5)^{n+1}}{n+1}\cdot\frac{n}{(2x-5)^n}\right| < 1 \Rightarrow -1 < 2x-5 < 1.$$

Testing endpoints gives: $2 \le x < 3$ Radius is $\frac{1}{2}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2x}{5}\right)^n$

$$\lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{5^{n+1}}\cdot\frac{5^n}{(2x)^n}\right| < 1 \Rightarrow -1 < \frac{2x}{5} < 1.$$

Testing endpoints gives: $-\frac{5}{2} < x < \frac{5}{2}$ Radius is $\frac{5}{2}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = \lim_{n\to\infty}\frac{x^2}{(2n+2)(2n+1)} = 0 < 1.$$

Therefore the interval of convergence is $-\infty < x < \infty$ Radius is infinite.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{x}{n+1} = 0 < 1.$$

Therefore the interval of convergence is $-\infty < x < \infty$ Radius is infinite.

4. Determine the Taylor Series of the function $f(x) = \ln x$ about $x = 1$.

as in review 1: $\displaystyle\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$

(a) What is the interval of convergence?
$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(n+1)}\cdot\frac{n}{(x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{n(x-1)}{n+1}\right| = |x-1| < 1.$
Testing the endpoints, the interval of convergence is $0 < x \leq 2$

(b) What is the radius of convergence?
Radius of convergence is 1.

(c) Write $\ln 2$ as an infinite series.
$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. This is the alternating harmonic series.

5. Determine the Taylor Series of the function $f(x) = \cos x$ about $x = 0$.
$f^0(0) = 1$ $f^1(0) = 0$ $f^2(0) = -1$ $f^3(0) = 0$
$f^4(0) = 1$ $f^5(0) = 0$ $f^6(0) = -1$ $f^7(0) = 0$
$f^8(0) = 1$ $f^9(0) = 0$ $f^10(0) = -1$ $f^11(0) = 0$
$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$

(a) What is the interval of convergence?
$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = \lim_{n\to\infty}\frac{x^2}{(2n+2)(2n+1)} = 0 < 1.$
Therefore the interval of convergence is $-\infty < x < \infty$

(b) What is the radius of convergence?
The radius of convergence is infinite.

6. Determine the Taylor Series of the function $f(x) = \sin x$ about $x = 0$.
$f^0(0) = 0$ $f^1(0) = 1$ $f^2(0) = 0$ $f^3(0) = -1$
$f^4(0) = 0$ $f^5(0) = 1$ $f^6(0) = 0$ $f^7(0) = -1$
$f^8(0) = 0$ $f^9(0) = 1$ $f^10(0) = 0$ $f^11(0) = -1$
$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$

7. Determine the Taylor Series of the function $f(x) = \sqrt{1+x}$ about $x = 0$.
$\sqrt{1+x} = 1+\frac{1}{2}x-\frac{1}{2^2\cdot 2!}x^2+\frac{1\cdot 3}{2^3\cdot 3!}x^3-\frac{1\cdot 3\cdot 5}{2^4\cdot 4!}x^4+\frac{3\cdot 5\cdot 7}{2^5\cdot 5!}x^5+... = \sum_{n=0}^{\infty}\binom{1/2}{n}x^n$
Look at:

$$f^0 = \sqrt{1+x} \qquad f^1 = \frac{1}{2\sqrt{1+x}} \qquad f^2 = \frac{1}{2}\cdot\frac{-1}{2(1+x)^{3/2}}$$

$$f^3 = \frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2(1+x)^{5/2}} \quad f^4 = \frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2(1+x)^{7/2}} \quad f^5 = \frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\cdot\frac{-7}{2(1+x)^{9/2}}$$

$$f^0(0) = 1 \qquad f^1(0) = \frac{1}{2} \qquad f^2(0) = -\frac{1}{2^2}$$

$$f^3(0) = \frac{1\cdot 3}{2^3} \qquad f^4(0) = -\frac{3\cdot 5}{2^4} \qquad f^5(0) = \frac{3\cdot 5\cdot 7}{2^5}$$

8. Determine the Taylor Series of the function $f(x) = \sqrt[3]{1+x}$ about $x = 0$.

9. Determine the Taylor Series of the function $f(x) = \sqrt[4]{1+x}$ about $x = 0$.

Answers to the Taylor Series for $f(x) = \sqrt[3]{1+x}$ and $f(x) = \sqrt[4]{1+x}$ follow in the same way the Taylor Series for $f(x) = \sqrt{1+x}$ does.