1. Determine $\frac{d\vec{T}}{ds}$ by following the given steps if $\vec{r}(t) =< \cos t, 2\sin t >$:

(a) Determine $\vec{r}\,'(t)$
Answer: $\vec{r}\,'(t) =< -\sin t, 2\cos t >$

(b) Determine $||\vec{r}\,'(t)||$
Answer: $||\vec{r}\,'(t)|| = \sqrt{\sin^2 t, +4\cos^2 t} = \sqrt{1+3\cos^2 t}$

(c) Determine $\vec{T}$ and $\frac{d\vec{T}}{dt}$
Answer: $\vec{T} =< \frac{-\sin t}{\sqrt{1+3\cos^2 t}}, \frac{2\cos t}{\sqrt{1+3\cos^2 t}} >$

$$\frac{d\vec{T}}{dt} = \left\langle \frac{-\sqrt{1+3\cos^2 t}\cos t - \sin t\frac{3\cos t\sin t}{\sqrt{1+\cos^2 t}}}{(1+3\cos^2 t)}, \frac{-2\sqrt{1+3\cos^2 t}\sin t + 2\cos t\frac{3\cos t\sin t}{\sqrt{1+\cos^2 t}}}{(1+3\cos^2 t)} \right\rangle$$

$$= \left\langle \frac{-(1+3\cos^2 t)\cos t - 3\sin^2 t\cos t}{(1+3\cos^2 t)^{3/2}}, \frac{-2(1+3\cos^2 t)\sin t + 6\cos^2 t\sin t}{(1+3\cos^2 t)^{3/2}} \right\rangle$$

$$= \left\langle \frac{-4\cos t}{(1+3\cos^2 t)^{3/2}}, \frac{-2\sin t}{(1+3\cos^2 t)^{3/2}} \right\rangle$$

(d) Determine $\frac{d\vec{T}}{ds} = \left(\frac{dt}{ds}\right)\frac{d\vec{T}}{dt}$

$$\frac{d\vec{T}}{ds} = \frac{1}{ds/dt}\frac{d\vec{T}}{dt} = \left\langle \frac{-4\cos t}{(1+3\cos^2 t)^2}, \frac{-2\sin t}{(1+3\cos^2 t)^2} \right\rangle$$

2. Evaluate the follow integrals as convergent or divergent. If convergent, calculate its value.

(a) $\int_0^\infty 7e^{-3x}\,dx$

$$= \frac{-7}{3}e^{-3x}\Big|_{x=0}^{x=t\to\infty} = \frac{7}{3}.$$

(b) $\int_5^\infty \frac{7\,dx}{(x-4)^2}$

$$= \frac{-7}{x-4}\Big|_{x=5}^{x=t\to\infty} = 7.$$

(c) $\int_3^4 \frac{7\,dx}{(x-3)^2}$

$$= \frac{-7}{x-3}\Big|_{x=t\to 3}^{4} = \infty.$$

(d) $\int_1^2 \frac{x}{\sqrt{x^2-1}}\,dx$

$$= \sqrt{x^2-1}\Big|_{x=t\to 1}^{2} = \sqrt{3}.$$

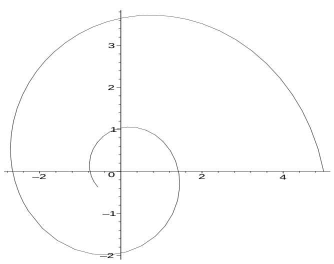
(e) $\int_1^\infty \frac{3x}{5+4x^2}\,dx$

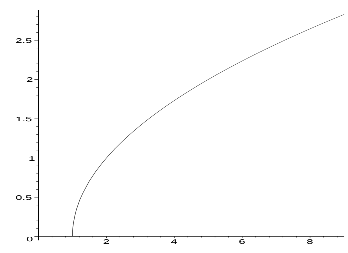
$$= \frac{3}{8}\ln\left(5+4x^2\right)\Big|_{x=1}^{x=t\to\infty} = \infty.$$

(f) $\int_0^\infty \frac{3x}{1+4x^4}\,dx$

$$= \frac{3}{4}\arctan\left(2x^2\right)\Big|_{x=0}^{x=t\to\infty} = \frac{3\pi}{8}.$$

3. If the spiral $< 5e^{-0.5t}\cos t, 5e^{-0.5t}\sin t >$ goes on forever, what will be the arclength? (This is an improper integral). (picture below)





$(x'(t))^2 = (-2.5e^{-0.5t}\cos t - 5e^{-0.5t}\sin t)^2 = 6.25e^{-t}\cos^2 t + 25\cos t\sin t + 25e^{-t}\sin^2 t$
$(y'(t))^2 = (-2.5e^{-0.5t}\sin t + 5e^{-0.5t}\cos t)^2 = 6.25e^{-t}\sin^2 t - 25\cos t\sin t + 25e^{-t}\cos^2 t$

$$\int_0^\infty \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^\infty \sqrt{31.25e^{-t}}\,dt = \int_0^\infty \sqrt{31.25}e^{-0.5t}\,dt = 2\sqrt{31.25}e^{-0.5t}\Big|_{t=o}^{t\to\infty}$$
$= 2\sqrt{31.25}.$

4. Determine the center of mass of the region with constant density and bounded by the x-axis, the line $x = 9$ and the function $f(x) = \sqrt{x-1}$. (picture above)
Area $= \int_1^9 \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2}\Big|_1^9 = 18.$

$$\overline{x} = \frac{M_y}{m} = \frac{\int_1^9 x\sqrt{x-1}\,dx}{18} = \frac{928\sqrt{2}/15}{18} = \frac{464\sqrt{2}}{135} \approx 4.86.$$

$$\overline{y} = \frac{M_x}{m} = \frac{\int_1^9 \frac{1}{2}(x-1)\,dx}{18} = \frac{16}{18} = \frac{8}{9}$$

5. Determine the mass and the center of mass of a 0.7-m rod whose density varies linearly from 3.0 kg/m to 3.7 kg/m. (Note: First determine the linear function of density.)
   $d(x) = x + 3$

$$\overline{x} = \frac{\int_0^{0.7} (x^2+3x)\,dx}{\int_0^{0.7} (x+3)\,dx} = \frac{637/750}{469/200} = \frac{364}{1005} \approx 0.3622m$$

6. Determine the Taylor Polynomial of the function $f(x) = e^x$ about $x = 0$.
   Consider: $e^x = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + ...$
   Then evaluating at $x = 0$ gives $e^0 = c_0$ so $c_0 = 1$. Differentiating both sides gives
   $e^x = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + 7c_7x^6 + ...$
   Then evaluating at $x = 0$ gives $e^0 = c_1$ so $c_1 = 1$ and differentiating both sides gives
   $e^x = 2c_2x + 3\cdot 2c_3x + 4\cdot 3c_4x^2 + 5\cdot 4c_5x^3 + 6\cdot 5c_6x^4 + 7\cdot 6c_7x^5 + 8\cdot 7c_8x^6 + 9\cdot 8c_9x^7 + ...$
   Evaluating at $x = 0$ gives $e^0 = 2c_2$ so $c_2 = \frac{1}{2}$. Differentiating both sides gives
   $e^x = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4x + 5\cdot 4\cdot 3c_5x^2 + 6\cdot 5\cdot 4c_6x^3 + 7\cdot 6\cdot 5c_7x^4 + 8\cdot 7\cdot 6c_8x^5 + 9\cdot 8\cdot 7c_9x^6 + ...$
   Evaluating at $x = 0$ gives $e^0 = 3\cdot 2c_3$ so $c_3 = \frac{1}{3\cdot 2}$. Differentiating both sides gives
   $e^x = 4\cdot 3\cdot 2c_4 + 5\cdot 4\cdot 3\cdot 2c_5x + 6\cdot 5\cdot 4\cdot 3c_6x^2 + 7\cdot 6\cdot 5\cdot 4c_7x^3 + 8\cdot 7\cdot 6\cdot 5c_8x^4 + 9\cdot 8\cdot 7\cdot 6c_9x^5 + ...$
   Evaluating at $x = 0$ gives $e^0 = 4\cdot 3\cdot 2c_4$ so $c_4 = \frac{1}{4\cdot 3\cdot 2}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

   (a) Write $e = e^1$ as an infinite series.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

   (b) Write $e^{-1} = \frac{1}{e}$ as an infinite series.

$$e^{-1} = \frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

(c) Write e^2 as an infinite series.

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!}$$

7. Determine the Taylor Polynomial of the function $f(x) = \ln x$ about $x = 1$.
A Taylor Polynomial for $f(x)$ about $x = 1$ can be written as:

$$\ln x = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!}(x-1)^n$$

$f^0(1) = \ln 1 = 0.$

$$f^1(x) = \tfrac{1}{x} \Rightarrow \tfrac{f^1(1)}{1!} = 1 \quad f^2(x) = \tfrac{-1}{x^2} \Rightarrow \tfrac{f^2(1)}{2!} = \tfrac{-1}{2} \quad f^3(x) = \tfrac{2}{x^3} \Rightarrow \tfrac{f^3(1)}{3!} = \tfrac{1}{3}$$
$$f^4(x) = \tfrac{-6}{x^4} \Rightarrow \tfrac{f^4(1)}{4!} = \tfrac{-1}{4} \quad f^5(x) = \tfrac{24}{x^5} \Rightarrow \tfrac{f^5(1)}{5!} = \tfrac{1}{5} \quad f^6(x) = \tfrac{-120}{x^6} \Rightarrow \tfrac{f^6(1)}{6!} = \tfrac{-1}{6}$$
$$f^7(x) = \tfrac{720}{x^7} \Rightarrow \tfrac{f^7(1)}{7!} = \tfrac{1}{7} \quad f^8(x) = \tfrac{-7(720)}{x^8} \Rightarrow \tfrac{f^8(1)}{8!} = \tfrac{-1}{8} \quad f^9(x) = \tfrac{56(720)}{x^9} \Rightarrow \tfrac{f^9(1)}{9!} = \tfrac{1}{9}$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n \qquad 0 < x \leq 2$$

(a) Write $\ln 0.5$ as an infinite series.

$$\ln 0.5 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{-1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{-1}{n \cdot 2^n}$$

(b) Write $\ln 1.5$ as an infinite series.

$$\ln 1.5 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot 2^n}$$

8. Determine the Taylor Polynomial of the function $f(x) = \dfrac{1}{1-x}$ about $x = 0$.
$f^0(0) = \frac{1}{1-0} = 1.$

$$f^1(x) = \tfrac{1}{(1-x)^2} \Rightarrow \tfrac{f^1(0)}{1!} = 1 \quad f^2(x) = \tfrac{2}{(1-x)^3} \Rightarrow \tfrac{f^2(0)}{2!} = 1 \quad f^3(x) = \tfrac{3!}{(1-x)^4} \Rightarrow \tfrac{f^3(0)}{3!} = 1$$
$$f^4(x) = \tfrac{4!}{(1-x)^5} \Rightarrow \tfrac{f^4(0)}{4!} = 1 \quad f^5(x) = \tfrac{5!}{(1-x)^6} \Rightarrow \tfrac{f^5(0)}{5!} = 1 \quad f^6(x) = \tfrac{6!}{(1-x)^7} \Rightarrow \tfrac{f^6(0)}{6!} = 1$$
$$f^7(x) = \tfrac{7!}{(1-x)^8} \Rightarrow \tfrac{f^7(0)}{7!} = 1 \quad f^8(x) = \tfrac{8!}{(1-x)^9} \Rightarrow \tfrac{f^8(0)}{8!} = 1 \quad f^9(x) = \tfrac{9!}{(1-x)^10} \Rightarrow \tfrac{f^9(0)}{9!} = 1$$

$$\frac{1}{x-1} = \sum_{n=0}^{\infty} x^n \qquad -1 < x < 1$$

(a) From your answer, what is $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$

Answer: $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{1-2/3} = 3.$

(b) From your answer, what is $\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n$

Answer: $\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n = \frac{1}{1-(-2/3)} = \frac{3}{5}.$

(c) From your answer, what is $\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n$

Answer: $\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n = \frac{1}{1-3/5} = \frac{5}{2}.$

9. Integrate the following:

(a) $\int 3xe^{-2x}\,dx = -\frac{3}{2}xe^{-2x} + \int \frac{3}{2}e^{-2x}\,dx = -\frac{3}{2}xe^{-2x} - \frac{3}{4}e^{-2x} + c$

$u = 3x \qquad dV = e^{-2x}\,dx$

$du = 3\,dx \qquad V = -\frac{1}{2}e^{-2x}$

(b) $\int 3xe^{-2x^2}\,dx = -\frac{3}{4}e^{-2x^2} + c$

Let $u = -2x^2$. Then $-\frac{1}{4}du = x\,dx$.

(c) $\int 3x^3e^{2x^2}\,dx = \frac{3}{4}x^2e^{2x^2} - \int \frac{3}{2}xe^{2x^2}\,dx = \frac{3}{4}x^2e^{2x^2} - \frac{3}{8}e^{2x^2} + c$

$u = 3x^2 \qquad dV = xe^{2x^2}\,dx$

$du = 6x\,dx \qquad V = \frac{1}{4}e^{2x^2}$

(d) $\int \frac{x+2}{2x^2-3x+1}\,dx = \int \left(-\frac{5}{2x-1} + \frac{3}{x-1}\right)dx = 3\ln|x-1| - 5\ln|2x-1| + c$

$\frac{x+2}{(2x-1)((x-1)} = \frac{A}{2x-1} + \frac{B}{x-1} \Rightarrow A(x-1) + B(2x-1) = x+2$

$x = 1 \Rightarrow B = 3; \qquad x = \frac{1}{2} \Rightarrow A = -5$

(e) $\int_{-5}^{5} \sqrt{25-x^2}\,dx$

area of a semicircle of radius $r = 5$ is $\frac{25\pi}{2}$

(f) $\int \frac{3e^t}{\sqrt{4-e^t}}\,dt = -6\sqrt{4-e^t} + c$

Let $u = 4 - e^t$. Then $-du = e^t\,dt$.