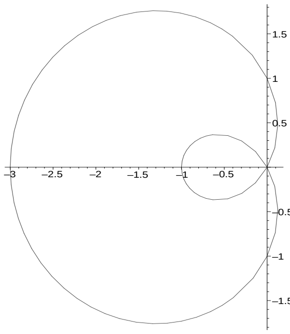
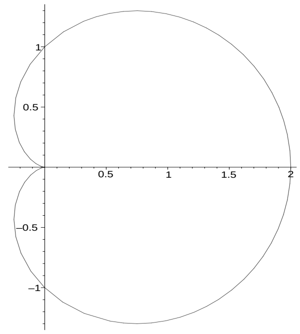
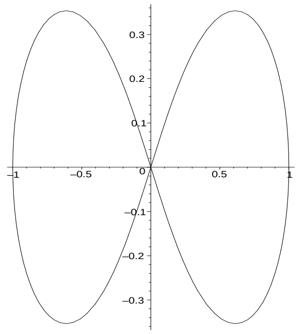
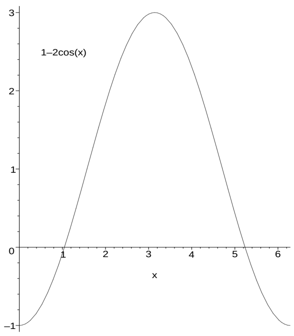
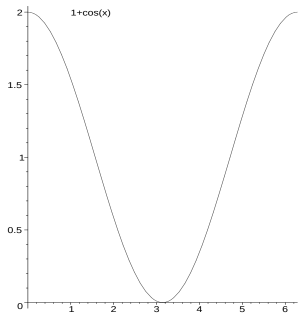
1. Match the polar curve with its graph and determine the area enclosed.
A) $r(\theta) = 1 + \cos\theta$ B) $r^2(\theta) = \cos 2\theta$ C) $r(\theta) = 1 - 2\cos\theta$

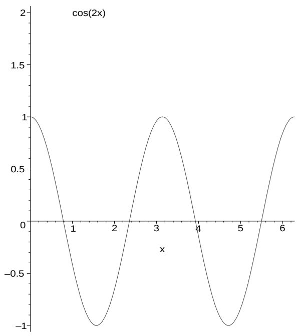












Answer: The first graph is $r(\theta) = 1 - 2\cos\theta$. Note that $r(\theta)$ is negative for $0 \le \theta < \pi/3$ and $5\pi/3 < \theta \le 2\pi$ and this forms the bottom of the inner loop as we start drawing and finishes with the the top of the inner loop.
The second graph is $r(\theta) = 1 + \cos\theta$. Note that $1 + \cos\theta$ is never negative and only zero at $\theta = \pi$.
The third graph is $r^2(\theta) = \cos 2\theta$. Note that $\cos 2\theta$ is negative for $\pi/4 < \theta < 3\pi/4$ and $5\pi/4 < \theta < 7\pi/4$ so $r(\theta)$ is imaginary. Therefore there is no graph within those intervals.
Areas:
For the first graph, $r(\theta) = 1 - 2\cos\theta$ ignore the inner loop. Then the area element is $dA = \dfrac{1}{2}(1 - 2\cos\theta)^2\, d\theta$ and $\dfrac{\pi}{3} \le \theta \le \dfrac{5\pi}{3}$ so that:

$$\int_{\pi/3}^{5\pi/3} dA = \int_{\pi/3}^{5\pi/3} \frac{1}{2}(1 - 4\cos\theta + 4\cos^2\theta)\, d\theta = \int_{\pi/3}^{5\pi/3} \frac{1}{2}(1 - 4\cos\theta + 2 + 2\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(3\theta - 4\sin\theta + \sin 2\theta\right)\Big|_{\theta=\pi/3}^{\theta=5\pi/3} = \frac{1}{2}\left(5\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right) = 2\pi + \frac{3\sqrt{3}}{2}.$$

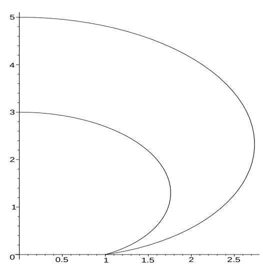
For the <u>second</u> graph, $r(\theta) = 1 + \cos\theta$, the area element is $dA = \frac{1}{2}(1+\cos\theta)^2\,d\theta$ and $0 \le \theta \le 2\pi$ so that:

$$\int_0^{2\pi} dA = \int_0^{2\pi} \frac{1}{2}(1 + 2\cos\theta + \cos^2\theta)\,d\theta = \int_0^{2\pi} \frac{1}{2}(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta)\,d\theta$$
$$= \left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big|_{\theta=0}^{\theta=2\pi} = 3\pi.$$

For the <u>third</u> graph, $r^2(\theta) = \cos 2\theta$, note that the two regions are symmetric so we'll find the area of one and double that value. To get the right loop, the area element is $dA = \frac{1}{2}(\cos 2\theta)^2\,d\theta$ and $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ so that:

$$2\int_{-\pi/4}^{\pi/4} dA = 2\int_{-\pi/4}^{\pi/4} \frac{1}{2}(\cos 2\theta)^2\,d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right)\,d\theta$$
$$= \left(\frac{1}{2}\theta + \frac{1}{8}\sin 4\theta\right)\Big|_{\theta=-\pi/4}^{\theta=\pi/4} = \frac{\pi}{4}.$$

2. Determine the area of the polar region bounded by the curves $r(\theta) = 1 + 2\sin\theta$ and $r(\theta) = 1 + 4\sin\theta$ and the line $\theta = \frac{\pi}{2}$.

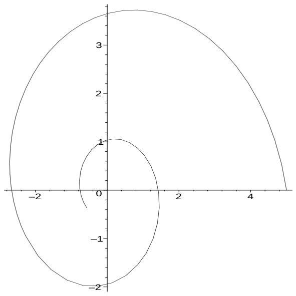


Answer: The outer area element is $dA_1 = \frac{1}{2}(1+4\sin\theta)^2\,d\theta$ and the inner area element is $dA_2 = \frac{1}{2}(1+2\sin\theta)^2\,d\theta$ so that a sector of $d\theta$ would have area
$dA = \frac{1}{2}\left[(1+4\sin\theta)^2 - (1+2\sin\theta)^2\right]\,d\theta$ so that:

$$\int_0^{\pi/2} dA = \int_0^{\pi/2} \frac{1}{2}(4\sin\theta + 12\sin^2\theta)\,d\theta = \int_0^{\pi/2} (2\sin\theta + 3 - 3\cos 2\theta)\,d\theta$$

$$= \left(-2\cos\theta + 3\theta - \frac{3}{2}\sin 2\theta\right)\Big|_{\theta=0}^{\theta=\pi/2} = \frac{3\pi}{2} + 2.$$

3. Determine the arclength of the spiral $< x(t), y(t) >=< e^{-0.5t}\cos(2t), e^{-0.5t}\sin(2t) >$ for which $0 \le \theta \le 5$.



Answer: Note that:

$$\begin{aligned}
(x'(t))^2 &= (-0.5e^{-0.5t}\cos(2t) - 2e^{-0.5t}\sin(2t))^2 \\
&= .25e^{-t}\cos^2(2t) + 2e^{-0.5t}\cos(2t)\sin(2t) + 4e^{-t}\sin^2(2t) \\
(y'(t))^2 &= (-0.5e^{-0.5t}\sin(2t) + 2e^{-0.5t}\cos(2t))^2 \\
&= .25e^{-t}\sin^2(2t) - 2e^{-0.5t}\cos(2t)\sin(2t) + 4e^{-t}\cos^2(2t) \\
\frac{ds}{dt} &= \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{4.25e^{-t}} = \sqrt{4.25}e^{-0.5t}
\end{aligned}$$

$$\int_0^5 \frac{ds}{dt}\,dt = \int_0^5 \sqrt{4.25}e^{-0.5t}\,dt = -2\sqrt{4.25}e^{-0.5t}\Big|_{t=0}^{t=5} = 2\sqrt{4.25}(1 - e^{-2.5}).$$

4. Note that in (1), the second graph is $r(\theta) = 1 + \cos\theta$ for $0 \le \theta \le 2\pi$. If one should walk this path, what is the distance traveled? (Tip: By symmetry, you could double the evaluation of the integral from 0 to π that would eliminate $\sqrt{\cos^2(t/2)} = |\cos(t/2)|$ that you would get when you apply the half-angle identity $1 + \cos 2t = 2\cos^2 t$.)

Answer: $\dfrac{ds}{d\theta} = \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2} = \sqrt{2 + 2\cos\theta} = 2|\cos(\theta/2)|$

$$\int_0^{\pi} \frac{ds}{d\theta}\,d\theta = 2\int_0^{\pi} 2\cos(\theta/2)\,d\theta = 8\sin(\theta)|_{\theta=0}^{\theta=\pi} = 8.$$

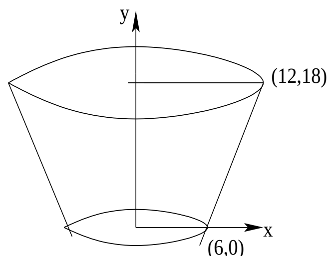
5. Determine the area of the region enclosed by the inner loop of the polar curve $r(\theta) = 1 - 2\cos\theta$ in the first graph of (1). (Hint: First determine the interval of θ

which determines the inner loop by finding the values of θ for which $r(\theta) = 0$.)
The interval for which the inner-loop is formed is $-\pi/3 \le \theta \le \pi/3$. So that:

$$\int_{-\pi/3}^{\pi/3} dA = \int_{-\pi/3}^{\pi/3} \frac{1}{2}(1 - 4\cos\theta + 4\cos^2\theta)\, d\theta = \int_{-\pi/3}^{\pi/3} \frac{1}{2}(1 - 4\cos\theta + 2 + 2\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}(3\theta - 4\sin\theta + \sin 2\theta)\Big|_{\theta=-\pi/3}^{\theta=\pi/3} = \frac{1}{2}\left(\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(-\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) = \pi - \frac{3\sqrt{3}}{2}.$$

6. Determine the work applied to empty a tank of water (62.4 lb/ft^3) from the top. The tank (in feet) is a truncated cone with bottom radius 6 ft, top radius 12, and height 18. See picture. (Hint: Determine the equation of the line connecting the two points shown.)



Answer: <u>Draw it and Label it!</u>
W = density $\times$ volume $\times$ distance
Consider a disk at a height of y with a thickness Δy. Its radius is the x-coordinate at y which is $x = \dfrac{y+18}{3}$ so that its volume is $V = \pi\left(\dfrac{y+18}{3}\right)^2 \Delta y \text{ft}^3$.
The density is 62.4lb/ft^3 and the distance up this disk has to be taken is $(18 - y)$ft.
Therefore, the work on a disk at y is $62.4\pi\left(\dfrac{y+18}{3}\right)^2 \Delta y(8-y)$ft-lb. Now take sum of the work elements so that:

$$W = \int_0^{18} 62.4\pi\left(\frac{y+18}{3}\right)^2 (18-y)\, dy = \frac{62.4\pi}{9}\int_0^{18}(324 - y^2)\, dy = \frac{(2)(62.4)\pi}{27}(18)^3.$$

7. An object moves from point $(-1, 0)$ to $(1, 0)$ on the graph of $y = x^3 - x$, (Note: $< x(t), y(t) >=< t, t^3 - t >$, acted on by a force that attracts it toward the point $(1, 0)$ and has magnitude proportional to the distance between the object and $(1, 0)$. Calculate the work done by the force. Note: $||\vec{F}|| = k\sqrt{(t-1)^2 + (t^3 - t)^2}$.
If $\vec{F}$ is in the direction of $(1, 0)$ from $< t, t^3 - t >$ then it is in the direction of

$< 1-t, t-t^3 >$ and $||\vec{F}||$ times a unit vector in this direction would be the force proportional to the distance between the object and $(1, 0)$.
Answer:
$\vec{F} = ||\vec{F}||\vec{F_u}$ where $\vec{F_u}$ is a unit vector in the direction of $\vec{F} =< 1-t, t-t^3 >$ so
$\vec{F_u} = \left\langle \dfrac{1-t}{\sqrt{(1-t)^2+(t-t^3)^2}}, \dfrac{t-t^3}{\sqrt{(1-t)^2+(t-t^3)^2}} \right\rangle$ so
$\vec{F} = ||\vec{F}||\vec{F_u} = k < 1-t, t-t^3 >$ and $d\vec{r} =< 1, 3t^2-1 >$. Therefore,

$$W = \int_{-1}^{1} \vec{F} \cdot d\vec{r} = k \int_{-1}^{1} (1-t-t+4t^3-3t^5)\, dt = 2k.$$

8. Integrate the following:

(a) $\displaystyle\int x^4 \ln x \, dx$

$u = \ln x \qquad dV = x^4 \, dx$

$du = \dfrac{1}{x} dx \qquad V = \frac{1}{5}x^5$

Answer: $\dfrac{1}{5}x^5 \ln x - \displaystyle\int \frac{1}{5}x^4 \, dx = \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + c.$

(b) $\displaystyle\int \frac{3e^t}{\sqrt{1+2e^t}} \, dt$

$u = 1 + 2e^t \qquad \dfrac{1}{2}du = e^t \, dt$

Answer: $\displaystyle\int \frac{3}{2}\frac{1}{\sqrt{u}} \, du = 3\sqrt{1+2e^t} + c.$

(c) $\displaystyle\int (\ln x)^2 \, dx$

$u = (\ln x)^2 \qquad dV = dx$

$du = \dfrac{2 \ln x}{x} \, dx \qquad V = x$

$x(\ln x)^2 - \displaystyle\int 2 \ln x \, dx$

$u = 2 \ln x \qquad dV = dx$

$du = \dfrac{2}{x} \, dx \qquad V = x$

Answer: $x(\ln x)^2 - 2x \ln x - \displaystyle\int 2dx = x(\ln x)^2 - 2x \ln x + 2x + c.$

(d) $\displaystyle\int \sin^2(3t) \, dt$

Answer: $\displaystyle\int \left(\frac{1}{2} - \frac{1}{2}\cos(6t)\right) \, dt = \frac{1}{2}t - \frac{1}{12}\sin(6t) + c.$

(e) $\int_0^4 \sqrt{16 - x^2}\, dx$

Answer: This is the integral over the quarter circle of radius 4. The integral value is the area value, 4π.

(f) $\int \frac{2x-3}{x^3 - 4x^2 + 4x}\, dx$

$\frac{2x-3}{x^3 - 4x^2 + 4x} = \frac{A}{x} + \frac{B}{(x-2)^2} + \frac{C}{x-2}$

$A(x-2)^2 + Bx + Cx(x-2) = 2x - 3$

$x = 2 \Rightarrow 2B = 1 \Rightarrow B = \frac{1}{2}$

$x = 0 \Rightarrow 4A = -3 \Rightarrow A = \frac{-3}{4}$

$Ax^2 + Cx^2 = 0x^2 \Rightarrow C = -A \Rightarrow C = \frac{3}{4}$

Answer: $\frac{-3}{4} \ln|x| - \frac{1}{2(x-3)} + \frac{3}{4} \ln|x-2| + c.$